

On finiteness of near-rings

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Prompted by a recent paper of PUTCHA and YAQUB [6], we resume the study of finiteness in near-rings, which was pursued by various authors several years ago (see [1], [3], [4] and [5]). Our results, most of which are for distributive near-rings, depend on the following results from [1]:

- (I) Every infinite distributive near-ring contains an infinite subring or an infinite subnear-ring with trivial multiplication.
- (II) Every infinite nil ring contains an infinite zero ring.

We deal with left near-rings N , denoting the additive group by N^+ , the centralizer of an element a in the group N^+ by $C_N(a)$, and the derived group of N^+ by N' . For $S \subseteq N$, the symbols $A_l(S)$, $A_r(S)$ and $A(S)$ denote the left, right, and two-sided annihilators of S ; and the symbol $\langle S \rangle$ denotes the subnear-ring generated by S . In one of our theorems we shall speak of FC-groups, defined as those in which each element has only finitely many conjugates, or equivalently as those in which the centralizer of each element has finite index. We shall also mention Tarski groups, defined as infinite groups in which every proper subgroup has order p , where p is an odd prime. The existence of Tarski groups has only recently been announced; and, as far as I am aware, a proof has not yet appeared in the literature.

1. Near-rings with finitely many non-nilpotent elements

Theorem 1. *If N is a non-nil distributive near-ring having only a finite number of non-nilpotent elements, then N is finite.*

PROOF. Note that N contains non-zero idempotents; indeed, each non-nilpotent element has an idempotent power. If there exists a non-zero central idempotent e , then for each nilpotent element u , $e+u$ is non-nilpotent; otherwise $e=e+u-u$ would be the difference of two commuting nilpotent elements, hence a nilpotent element. Thus $\{e+u \mid u \text{ nilpotent}\}$ is finite; hence N has only finitely many nilpotent elements, so N is finite. In particular, N is finite if it has a multiplicative identity.

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We make our proof by induction on the number $n(N)$ on non-zero idempotents. If $n(N)=1$, let e be the unique non-zero idempotent and let x be any element of N . Then $e+xe-exe$ and $e+ex-exe$ are both non-zero idempotents, hence both equal to e . It follows that e is central, hence N is finite.

Now suppose that our theorem is true for all N_1 with $n(N_1)<k$, and consider N with $n(N)=k$. Let e be a non-zero idempotent, which we may assume to be non-central, and write

$$(1) \quad N = Ne + A_1(e).$$

If $A_1(e)$ is non-nil, then $n(Ne)<k$ and $n(A_1(e))<k$, hence finiteness of N follows by the inductive hypothesis; thus, we assume $A_1(e)$ is nil and let A be any subnear-ring of $A_1(e)$ with trivial multiplication. Then for $u \in A$ we see that $(e+u)^2 = e+eu$ is a non-zero idempotent, hence $e+u$ is not nilpotent. Thus, $\{e+u | u \in A\}$ is finite, which forces A to be finite; and an appeal to (I) and (II) shows that $A_1(e)$ is finite. Now write $Ne = eNe + (Ne \cap A_r(e))$, and repeat the above argument to show the second summand is finite. But eNe is finite because it has a multiplicative identity; hence Ne is finite, and by (1), N is finite. This completes the induction.

Applying this theorem yields an extension of Theorems 2.1 and 2.2 of [5]; the proof is omitted, since it is the same as the proof of the ring-theoretic version in [2].

Theorem 2. *Let N be a distributive near-ring having only a finite number $n \geq 1$ of non-nilpotent zero divisors. Then N is finite.*

As past experience predicts, it is difficult to extend these results even to distributively-generated near-rings; however, our next theorem is one such extension. Recall that a near-ring N is zero-commutative if $ab=0$ implies $ba=0$, and N is an IFP near-ring if $ab=0$ implies $axb=0$ for all $x \in N$.

Theorem 3. *Let N be a non-nil, distributively-generated, and zero-commutative near-ring. If N has only finitely many non-nilpotent elements, then N is finite.*

PROOF. Let D be any set of distributive elements generating N^+ , and let $-D = \{-d | d \in D\}$. Note that if $d_1, \dots, d_k$ are in $D \cup (-D)$ and n is a positive integer, then $(d_1 + d_2 + \dots + d_k)^n$ is a sum of terms each of which is a product of n of the d_i . Now the hypothesis that N is zero-commutative implies that N is an IFP near-ring; hence if $d_1, \dots, d_k$ are nilpotent elements belonging to $D \cup (-D)$, $d_1 + \dots + d_k$ is also nilpotent. Thus, our hypothesis that N is non-nil guarantees that D contains a non-nilpotent element, hence a non-zero idempotent e . Since N is zero-commutative, e must be central.

Let u be an arbitrary nilpotent element of N . If we can show that $e+u$ is non-nilpotent, then finiteness of N follows as in the first paragraph of the proof of Theorem 1. Now if $e+u$ were nilpotent, then e would be of the form $u_1 - u_2$, with u_1 and u_2 nilpotent and $u_1 - u_2$ central. However, for any pair $u_1, u_2 \in N$ with $u_1 - u_2$ central, we can show by induction on n that $(u_1 - u_2)^n$ is a sum of terms of form $\pm v_1 v_2 \dots v_n$, where each v_i is either u_1 or u_2 ; and it follows by IFP that if u_1 and u_2 are nilpotent, then so is $u_1 - u_2$. Thus, our proof is complete.

2. Near-rings with chain conditions on non-nil subnear-rings

In [2], there appears the following result, which we shall refer to as Proposition R_4 : if the ring R is not a nil ring and has both ascending chain condition and descending chain condition on non-nil subrings, then R is finite. We shall not attempt to prove the corresponding result for distributive near-rings, for — assuming that Tarski groups do in fact exist — the direct sum of a field $GF(p)$ and a trivial near-ring on a Tarski group is a counterexample. However, under suitable restrictions on the additive group, Proposition R_4 does generalize to distributive near-rings.

Before proceeding, we note that if N is any distributive near-ring, N^2 is a ring, and N' is an ideal contained in $A(N)$.

Theorem 4. *Let N be a non-nil distributive near-ring having both ascending chain condition and descending chain condition on non-nil subnear-rings. If N^+ is an FC-group, then N is finite.*

PROOF. The factor near-ring $\bar{N} = N/N'$ is a ring; and since $N' \subseteq A(N)$, $\bar{N}$ inherits the hypotheses of Theorem 4, hence by Proposition R_4 must be finite. It follows that N is periodic, and must therefore contain non-zero idempotents. If it happens that N contains a regular idempotent, then N has 1 and is consequently a ring; thus, we may assume that N contains a non-zero idempotent e which is a left zero divisor, and we may write

$$(2) \quad N = eN + A_r(e).$$

Assume that N is infinite. Since N^2 is a ring, it is necessarily finite; and since $e^2 + xy - e^2 - xy = 0$ for all $x, y \in N$, we have $eN \subseteq N^2 \subseteq C_N(e)$. The additive group $C_N(e)$ must be infinite, since it is of finite index in N^+ ; hence, (2) and the finiteness of eN imply that $S = C_N(e) \cap A_r(e)$ is infinite. In fact, the observation that $N^2 \subseteq C_N(e)$ guarantees that S is a subnear-ring; and applying (I) shows that S contains either an infinite near-ring with trivial multiplication or an infinite ring having a.c.c. and d.c.c. on non-nil subrings. In the latter case, (II) and Proposition R_4 imply that S contains an infinite zero ring.

Let B be an infinite subnear-ring of S with trivial multiplication. Now Be , being contained in N^2 , must be finite; hence, by the first isomorphism theorem for groups, $B_1 = B \cap A_l(e)$ must be infinite. Moreover, each element of B_1 is of finite additive order, for if $u \in B_1$ had infinite order, u and e would generate a counterexample to Proposition R_4 . Since B_1^+ is an FC-group, we see that for each finite subset $\{u_1, \dots, u_k\}$ of B_1 , $\bigcap_{i=1}^k B_1 \cap C_N(u_i)$ has finite index in B_1 , hence is infinite; therefore, we can construct inductively an infinite sequence $u_1, u_2, \dots$ of pairwise-additively-commutative elements of B_1 , which together with e generate an infinite ring — a counterexample to Proposition R_4 . Thus, the assumption that N is infinite must be false.

Theorem 5. *Let N be a non-nil distributive near-ring with ascending chain condition and descending chain condition on non-nil subnear-rings. If N^+ is nilpotent, then N is finite.*

PROOF. Let $N = N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots \supseteq N_r = \{0\}$ be the lower central series of N^+ ; and let m be the smallest positive integer for which N_m is finite. If $m=1$, there is nothing to prove. If $m=2$, N' is finite; and since N/N' is finite by Proposition R_4 , N must be finite. Thus, we need only consider the case $m \geq 3$.

Let e be a non-zero idempotent, the existence of which follows from the initial steps of the proof of Theorem 4. Since N is periodic, e must have finite additive order; and we now show that elements of N_{m-1} also have finite additive order. Accordingly, for fixed $u \in N_{m-1}$, consider the descending chain $\langle e, u \rangle \supseteq \langle e, 2u \rangle \supseteq \langle e, 4u \rangle \supseteq \dots$. Since it must become stationary at some point, there exists k such that $2^k u \in \langle e, 2^{k+1} u \rangle$; and since $N_{m-1} \subseteq N' \subseteq A(N)$, we have integers h, j and an element $c_1 \in [N_{m-1}, N] = N_m$ for which $2^k u = he + j2^{k+1} u + c_1$. Left-multiplying this equation by e yields $he = 0$, hence

$$(3) \quad 2^k u = j2^{k+1} u + c_1.$$

Substituting (3) into itself, we get $2^k u = 2j(2^k u) + c_1 = 2j(j2^{k+1} u + c_1) + c_1 = (2j)^2 2^k u + c_2$ for some $c_2 \in N_m$; and continuing inductively, we obtain a sequence $c_1, c_2, c_3, \dots$ of elements of N_m such that $2^k u = (2j)^s 2^k u + c_s$ for $s = 1, 2, \dots$. The finiteness of N_m yields distinct positive integers p, q for which $c_p = c_q$, and therefore $((2j)^p - (2j)^q)2^k u = 0$; thus, provided $j \neq 0$, u is now seen to have finite additive order. But if $j = 0$, then $2^k u \in N_m$; and again the finiteness of N_m guarantees that $2^k u$, and hence u , has finite order.

Consider the collection $\mathcal{C}$ of all subnear-rings of form $\langle e, u_1, \dots, u_k \rangle$, where $u_1, \dots, u_k \in N_{m-1}$. The ascending chain condition on non-nil subnear-rings yields a maximal member of $\mathcal{C}$, say E , which clearly contains N_{m-1} . But $E = \langle e, u_1, \dots, u_k \rangle$ is generated as an additive group by $e, u_1, \dots, u_k$; and in a nilpotent group, any subgroup generated by finitely many elements of finite order must be finite. Thus, N_{m-1} is finite, contrary to the minimality of m ; consequently, the case $m \geq 3$ cannot occur, and N is finite.

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