

Lie structure in prime rings with derivations

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Abstract

Recently J. BERGEN, I. N. HERSTEIN and J. W. KERR [2, Theorem 4] proved that if R is a prime ring, $\text{char } R \neq 2$, and $U \subseteq Z$ (the centre of R) is a Lie ideal of R ; further if δ and d are derivations of R such that $\delta d(U) = 0$, then either $\delta = 0$ or $d = 0$. A remarkable theorem of HERSTEIN [4, Theorem 1] of which we have made several uses states: Let R be a semiprime, 2-torsion free ring and let U be a Lie ideal of R . If $t \in R$ commutes with all $tu - ut$, $u \in U$, then $[t, U] = 0$. Further, AWTAR [1, Theorems 1 and 2] proved an interesting result which can be mentioned as follows: Let R be a prime ring, $\text{char } R \neq 2$, and let U be a Lie ideal of R . Let d be a nonzero derivation of R such that $ud(u) - d(u)u \in Z$ for all $u \in U$. Moreover, if $\text{char } R \neq 3$ or $u^2 \in U$ for all $u \in U$ then $U \subseteq Z$. The object of this paper is to extend the above mentioned results, to a more general situation.

Throughout this paper by ring R we mean an associative ring. The symbol Z denotes the centre of R . We say that a ring R is semiprime if it contains no nonzero nilpotent ideals, and R is prime if the nonzero elements of Z are not zero divisors in R . For $x, y \in R$, we denote $[x, y] = xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$, $r \in R$. For any subset A of R , we denote the centralizer of A by $C_R(A)$, and define as follows: $C_R(A) = \{r \in R \mid [a, r] = 0 \text{ for all } a \in A\}$. An additive mapping d from R into itself is said to be a derivation of R if $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$.

We begin this paper with the following theorem which may have some independent interest.

Theorem 1. *Let R be a prime ring, $\text{char } R \neq 2$, and let $d \neq 0$ be a derivation of R . Suppose that $U \not\subseteq Z$ is a Lie ideal of R such that $d^2(U) \subseteq Z$, and if $a \in R$ is such that $d(a) = 0$ and $[a, d(U)] \subseteq Z$. Then $a \in Z$.*

PROOF. By hypothesis, $d^2(u) \in Z$ for all $u \in U$. If $u \in U$, $r \in R$, then $d^2[u, r] \in Z$ and so $[d^2(u), r] + 2[d(u), d(r)] + [u, d^2(r)] \in Z$. Thus, since $d^2(u) \in Z$ for $u \in U$, we get

$$(1) \quad 2[d(u), d(r)] + [u, d^2(r)] \in Z \text{ for all } u \in U, r \in R.$$

Replace r by $rd^2(v)$ where $v \in U$ in (1) and expand; then

$$2[d(u), d(r)d^2(v) + rd^3(v)] + [u, d^2(r)d^2(v) + 2d(r)d^3(v) + rd^4(v)] \in Z,$$

or,

$$2[d(u), d(r)]d^2(v) + 2[d(u), r]d^3(v) + [u, d^2(r)]d^2(v) + 2[u, d(r)]d^3(v) + [u, r]d^4(v) \in Z,$$

since $d^2(v), d^3(v)$ and $d^4(v)$ are in Z . From (1) $2[d(u), d(r)] + [u, d^2(r)] \in Z$ and by hypothesis $d^2(v) \in Z$, therefore $2[d(u), d(r)]d^2(v) + [u, d^2(r)]d^2(v) \in Z$. Hence from the last equation, we get that

$$(2) \quad 2[d(u), r]d^3(v) + 2[u, d(r)]d^3(v) + [u, r]d^4(v) \in Z \text{ for all } u, v \in U \text{ and } r \in R.$$

In (2) replace r by $d(w)$ where $w \in U$ to get $2[d(u), d(w)]d^3(v) + [u, d(w)]d^4(v) \in Z$. Write $r = w$, $w \in U$, in (1) then $2[d(u), d(w)] \in Z$ and by hypothesis $d^3(v) \in Z$, therefore $2[d(u), d(w)]d^3(v) \in Z$. Thus we conclude that $[u, d(w)]d^4(v) \in Z$ for all $u, v, w \in U$. If $d^4(v) \neq 0$ for some $v \in U$, since $d^4(v) \in Z$ and R is prime, then $[u, d(w)] \in Z$ and so $[d(w), [d(w), u]] = 0$ for all $u, w \in U$. By Theorem 1 of [4], $[d(w), u] = 0$ for all $u, w \in U$; that is $d(U) \subset C_R(U) = Z$ by [2, Lemma 2] and so $U \subset Z$ by [2, Lemma 6]; a contradiction. Hence $d^4(U) = 0$. Thus from (2) we get $2[d(u), r]d^3(v) + 2[u, d(r)]d^3(v) \in Z$; that is $\{[d(u), r] + [u, d(r)]\}d^3(v) \in Z$ and so $d[u, r]d^3(v) \in Z$ for $u, v \in U, r \in R$. If $d^3(v) \neq 0$ for some $v \in U$, then $d[u, r] \in Z$ for $u \in U, r \in R$ since $d^3(v) \in Z$ and R is prime. Let $r = rd^2(w)$ where $w \in U$; we get, since $d^2(w) \in Z$, that $d([u, r]d^2(w)) \in Z$ and so $d[u, r]d^2(w) + [u, r]d^3(w) \in Z$. Since $d[u, r] \in Z$ and $d^2(w) \in Z$, then $d[u, r]d^2(w) \in Z$. Thus $[u, r]d^3(w) \in Z$ for all $w \in U, r \in R$. As $d^3(v) (\neq 0) \in Z$ and R is prime, we force that $[u, r] \in Z$; that is $[u, [u, r]] = 0$ for $u \in R, r \in R$ and so $U \subset Z$, by lemma 1.1.9 of [3], a contradiction. Hence $d^3(U) = 0$.

In (1) write $r = ra$, since $d(a) = 0$, we get on expansion

$$2[d(u), d(r)]a + 2d(r)[d(u), a] + [u, d^2(r)]a + d^2(r)[u, a] \in Z,$$

or,

$$\{2[d(u), d(r)] + [u, d^2(r)]\}a + 2d(r)[d(u), a] + d^2(r)[u, a] \in Z.$$

Commuting the last equation on x where $x \in R$, since by (1) $2[d(u), d(r)] + [u, d^2(r)] \in Z$ and by hypothesis $[a, d(u)] \in Z$, we get

$$\{2[d(u), d(r)] + [u, d^2(r)]\}[a, x] + 2[d(r), x][d(u), a] + [d^2(r), x][u, a]$$

$$(3) \quad + d^2(r)[[u, a], x] = 0 \text{ for } u \in U; r, x \in R.$$

Replace r by ar in (3) and use (3), since $d(a) = 0$, then

$$\{2[d(u), a]d(r) + [u, a]d^2(r)\}[a, x] + 2[a, x]d(r)[d(u), a] + [a, x]d^2(r)[u, a] = 0$$

$$(4) \quad \text{for } u \in U; r, x \in R.$$

Replace r by $d(v)$ in (4) where $v \in U$; since $d^3(U) = 0$, $d^2(U) \subset Z$ and $[a, d(U)] \subset Z$ we get $4d^2(v)[d(u), a][a, x] = 0$ and so $d^2(v)[d(u), a][a, x] = 0$ for $u, v \in U, x \in R$. Let $x = xy$ where $y \in R$; we get that $d^2(v)[d(u), a]R[a, x] = 0$. Since R is prime, then either $a \in Z$ or $d^2(v)[d(u), a] = 0$ for $u, v \in U$. If $d^2(U) = 0$, since $d \neq 0$, then by Theorem 1 of [2] $U \subset Z$; a contradiction. So $d^2(U) \neq 0$ therefore the above relation gives us $[d(u), a] = 0$ for all $u \in U$. By Theorem 2 of [2], since $U \not\subset Z$ and $d \neq 0$, $a \in Z$. Hence $a \in Z$. This proves the Theorem.

An immediate consequence of Theorem 1 is the following

Theorem 2. Let R be a prime ring, $\text{char } R \neq 2$, and let U a Lie ideal of R . Suppose that $a \in R$ is such that $[a, [a, u]] \in Z$ for all $u \in U$. Then $[a, U] = 0$. Moreover, if $U \not\subset Z$ then $a \in Z$.

PROOF. For $x \in R$, let $d(x) = [a, x]$ be an inner derivation of R . Then $d(a) = 0$. By hypothesis $[a, d(u)] \in Z$ and $d^2(u) \in Z$ for all $u \in U$. If for some $u \in U$, $[a, u] \neq 0$ then $d \neq 0$ and $U \not\subseteq Z$. Hence, by Theorem 1 $a \in Z$, a contradiction. Thus $[a, U] = (0)$. If $U \subseteq Z$, since $[a, U] = 0$; that is $a \in C_R(U)$, then by lemma 2 of [2] $C_R(U) = Z$ and so $a \in Z$.

We are in position to prove the following theorem which extends some due to HERSTEIN [4, Theorem 1].

Theorem 3. *Let R be a semiprime, 2-torsion free ring and let U be a Lie ideal of R . Suppose that $a \in R$ is such that $[a, [a, u]] \in Z$ for all $u \in U$. Then $[a, U] = 0$.*

PROOF. Since R is semiprime, then $\bigcap_{i \in I} P_i = (0)$ where I is an indexed set and P_i is a prime ideal of R . Thus $R_i = R/P_i$ is a prime ring of characteristic not equal to 2, since R is 2-torsion free; and $U_i = U/P_i$ is a Lie ideal of R_i . Since $[a, [a, u]] \in Z$ for all $u \in U$, then $[\bar{a}, [\bar{a}, u_i]] \in Z_i$ (the centre of R_i) for all $u_i \in U_i$ where $\bar{a} = a + P_i \in R_i$. By Theorem 2, $[\bar{a}, u_i] = 0$ for all $u_i \in U_i$ and so $[a, U] \subseteq \bigcap_{i \in I} P_i = (0)$.

With this theorem 3 is proved.

We are now proving the key theorem of this paper which generalizes Theorem 2 of [2].

Theorem 4. *Let R be a prime ring, $\text{char } R \neq 2$, and let $U \not\subseteq Z$ be a Lie ideal of R . If $a \in R$ is such that $[a, d(u)] \in Z$ for all $u \in U$, where $d \neq 0$ is a derivation of R , then $a \in Z$.*

PROOF. Suppose on the contrary that $a \notin Z$. We claim that $d(a) \in Z$. This we prove in the same way as in [2]. Let $V = [U, U] \subseteq U$ be a Lie ideal of R . It is clear that, if $v \in V$ then $d(v) \in U$. By hypothesis $[a, d(v)] \in Z$ for all $v \in V$. Then, $d[a, d(v)] \in Z$ and so $[d(a), d(v)] + [a, d^2(v)] \in Z$. But $[a, d^2(v)] \in Z$ for all $v \in V$, since $d(v) \in U$; therefore $[d(a), d(v)] \in Z$ for all $v \in V$. Replace v by $[a, v]$, then $[d(a), d[a, v]] \in Z$ and so $[d(a), [d(a), v]] + [a, d(v)] \in Z$. Since $[a, d(v)] \in Z$, we conclude that $[d(a), [d(a), v]] \in Z$ for all $v \in V$. Since $U \not\subseteq Z$, $V \not\subseteq Z$ by lemma 1 of [4]. By Theorem 2, $d(a) \in Z$.

For $r \in R$, let $\alpha(r) = [a, r]$ be an inner derivation of R . Then by hypothesis $\alpha d(u) \in Z$ for $u \in U$. For $u \in U$, $r \in R$ it follows that $[u, r] \in U$ and so $\alpha d[u, r] \in Z$ which yields on expansion

$$[\alpha d(u), r] + [\alpha(u), d(r)] + [d(u), \alpha(r)] + [u, \alpha d(r)] \in Z,$$

or,

$$(1) \quad [\alpha(u), d(r)] + [d(u), \alpha(r)] + [u, \alpha d(r)] \in Z \quad \text{for all } u \in U, r \in R.$$

Replace r by xy where $x, y \in R$ in (1) to get

$$(2) \quad \begin{aligned} & [\alpha(u), d(x)y + xd(y)] + [d(u), \alpha(x)y + x\alpha(y)] + [u, \alpha d(x)y + \alpha(x)d(y) + \\ & + d(x)\alpha(y) + x\alpha d(y)] \in Z \quad \text{for } x, y \in R, u \in U. \end{aligned}$$

In (2) write $x = y = a$, since $\alpha(a) = 0$ and $d(a) \in Z$ so $\alpha d(a) = 0$, then $2[\alpha(u), \alpha d(a)] \in Z$; that is $[\alpha(u), a]d(a) \in Z$. Thus $[a, [a, u]]d(a) \in Z$ for all $u \in U$.

If $d(a) \neq 0$, since R is prime and $d(a) \in Z$, we conclude that $[a, [a, u]] \in Z$ for all $u \in U$. By Theorem 2 $a \in Z$; a contradiction. Hence $d(a) = 0$.

Replace y by a in (2), since $d(a) = \alpha(a) = 0$, we get

$$[\alpha(u), d(x)a] + [d(u), \alpha(x)a] + [u, \alpha d(x)a] \in Z,$$

or,

$$\{[\alpha(u), d(x)] + [d(u), \alpha(x)] + [u, \alpha d(x)]\}a + d(x)[\alpha(u), a] + \alpha(x)[d(u), a] + \alpha d(x)[u, a] \in Z,$$

or,

$$(3) \quad \{[\alpha(u), d(x)] + [d(u), \alpha(x)] + [u, \alpha d(x)]\}a - d(x)\alpha^2(u) - \alpha(x)\alpha d(u) - \alpha d(x)\alpha(u) \in Z$$

for all $u \in U, x \in R$.

Commute (3) with a , as $\alpha d(u) \in Z$ and by (1) $[\alpha(u), d(x)] + [d(u), \alpha(x)] + [u, \alpha d(x)] \in Z$, to get

$$[d(x), a]\alpha^2(u) + d(x)[\alpha^2(u), a] + [\alpha(x), a]\alpha d(u) + [\alpha d(x), a]\alpha(u) + \alpha d(x)[\alpha(u), a] = 0$$

or,

$$(4) \quad \alpha d(x)\alpha^2(u) + d(x)\alpha^3(u) + \alpha^2(x)\alpha d(u) + \alpha^2 d(x)\alpha(u) + \alpha d(x)\alpha^2(u) = 0$$

for all $u \in U, x \in R$.

Replace u by $d(v)$ in (4) where $v \in V$, as $\alpha d(v) \in Z$ so $\alpha^2 d(v) = 0$, then

$$(5) \quad \alpha^2(x)\alpha d^2(v) + \alpha^2 d(x)\alpha d(v) = 0 \quad \text{for all } x \in R, v \in V.$$

In (5) replace x by w where $w \in U$, then $\alpha^2(w)\alpha d^2(v) = 0$ since $\alpha^2 d(w) = 0$. As $d(v) \in U$, $\alpha d^2(v) \in Z$ for all $v \in V$. If $\alpha d^2(v) \neq 0$ for some $v \in V$, then $\alpha^2(U) = 0$ since R is prime. Thus $[a, [a, u]] = 0$ for all $u \in U$ and so by Theorem 2 we force that $a \in Z$; a contradiction. Hence $\alpha d^2(v) = 0$ for all $v \in V$. Therefore, by (5), $\alpha^2 d(x)\alpha d(v) = 0$ for all $x \in R, v \in V$. If $\alpha d(v) = [a, d(v)] = 0$ for all $v \in V$, since $V \not\subset Z$ and $d \neq 0$, by Theorem 2 of [2] we conclude that $a \in Z$, a contradiction. Thus $\alpha d(v) \neq 0$ and so from the above we get $\alpha^2 d(x) = 0$ for $x \in R$, since R is prime and $\alpha d(v) \in Z$. Replace x by $d(x)d(v)$ where $v \in V$, then

$$\begin{aligned} 0 &= \alpha^2 d(d(x)d(v)) = \alpha^2 \{d^2(x)d(v) + d(x)d^2(v)\} = \\ &= \alpha^2 d^2(x)d(v) + 2\alpha d^2(x)\alpha d(v) + d^2(x)\alpha^2 d(v) + \alpha^2 d(x)d^2(v) + 2\alpha d(x)\alpha d^2(v) + d(x)\alpha^2 d^2(v). \end{aligned}$$

As we have seen above that $\alpha^2 d(R) = 0$ and $\alpha d^2(v) = 0$, therefore from the last equation we get $2\alpha d^2(x)\alpha d(v) = 0$ and so $\alpha d^2(x)\alpha d(v) = 0$ for all $x \in R, v \in V$. We have seen above that $\alpha d(v) \neq 0$, and as it is in the center Z , so $\alpha d^2(R) = 0$. Now, for $x \in R$, $\alpha d(x) = [a, d(x)] = d[a, x] = d\alpha(x)$. Thus $\alpha d = d\alpha$. Therefore, $\alpha d\alpha = d\alpha^2 = \alpha^2 d = 0$ and $d\alpha d = d^2\alpha = \alpha d^2 = 0$.

Since $\alpha^2 d = 0$, therefore from (4) we get

$$(6) \quad 2\alpha d(x)\alpha^2(u) + d(x)\alpha^3(u) + \alpha^2(x)\alpha d(u) = 0 \quad \text{for } u \in U, x \in R.$$

Replace u by $\alpha(u) = [a, u]$ and x by $v, v \in U$ in (3), since $\alpha d\alpha = 0$ and $\alpha d(U) = d\alpha(U) \subset Z$, we get

$$[\alpha^2(u), d(v)]a - d(v)\alpha^3(u) - \alpha d(v)\alpha^2(u) \in Z \quad \text{for all } u, v \in U.$$

In (6) replace x by v , where $v \in U$ and then adding to the last equation to get

$$(7) \quad [\alpha^2(u), d(v)]a + \alpha d(v)\alpha^2(u) + \alpha^2(v)\alpha d(u) \in Z \quad \text{for } u, v \in U.$$

Replace x by $\alpha(v)$ in (3) where $v \in U$, as $\alpha d\alpha = 0$ and $d\alpha(v) = \alpha d(v) \in Z$,

$$(8) \quad [d(u), \alpha^2(v)]a - d\alpha(v)\alpha^2(u) - \alpha^2(v)\alpha d(u) \in Z \quad \text{for } u, v \in U.$$

Adding (7) and (8) to get

$$(9) \quad \{[\alpha^2(u), d(v)] + [d(u), \alpha^2(v)]\}a \in Z \quad \text{for all } u, v \in U.$$

Let $r = \alpha(v)$, $v \in U$ in (1); we get, since $\alpha d\alpha = 0$ and $d\alpha(v) = \alpha d(v) \in Z$, that $[d(u), \alpha^2(v)] \in Z$. Replace r by v , $v \in U$ and u by $\alpha(u) = [a, u]$ in (1), then $[\alpha^2(u), d(v)] \in Z$. After adding these, we have $\beta = [\alpha^2(u), d(v)] + [d(u), \alpha^2(v)] \in Z$. If $\beta \neq 0$, then in view of (9) we get $a \in Z$; a contradiction. Thus $\beta = 0$; so

$$(10) \quad [\alpha^2(u), d(v)] + [d(u), \alpha^2(v)] = 0 \quad \text{for all } u, v \in U.$$

Replace u by $\alpha(u)$ in (10), then we get $[\alpha^3(u), d(v)] = 0$ for all $u, v \in U$. By Theorem 2 of [2], $\alpha^3(U) \subset Z$. Let $x = d(x)$ in (6); we get, since $\alpha d^2 = \alpha^2 d = 0$, that $d^2(x)\alpha^3(u) = 0$ for $x \in R$, $u \in U$. If $\alpha^3(U) \neq 0$, as it is in the centre Z , $d^2 = 0$. However, as proof of Lemma 1.1.9 of [3] shows, if R is a semiprime, 2-torsion free ring and d is a derivation of R such that $d^2 = 0$ then $d = 0$. Hence $\alpha^3(U) = 0$.

Putting $u = d(w)$ in (10) where $w \in V$, since $\alpha^2 d = 0$, then

$$(11) \quad [d^2(w), \alpha^2(v)] = 0 \quad \text{for } v \in U, w \in V.$$

Replace x by $\alpha^2(v)$, $v \in U$ and u by $d(w)$, $w \in V$ in (2), since $d\alpha^2 = 0$, $\alpha^3(U) = 0$ and $\alpha d(w) \in Z$, we have

$$[d^2(w), \alpha^2(v)\alpha(y)] + [d(w), \alpha^2(v)\alpha d(y)] \in Z,$$

or,

$$[d^2(w), \alpha^2(v)]\alpha(y) + \alpha^2(v)[d^2(w), \alpha(y)] + [d(w), \alpha^2(v)]\alpha d(y) + \alpha^2(v)[d(w), \alpha d(y)] \in Z \quad \text{for } v \in U, w \in V, y \in R.$$

In view of (11) the last equation reduces to

$$(12) \quad \gamma = \alpha^2(v)[d^2(w), \alpha(y)] + [d(w), \alpha^2(v)]\alpha d(y) + \alpha^2(v)[d(w), \alpha d(y)] \in Z$$

for all $v \in U, w \in V$ and $y \in R$.

Let $y = ya$ in (12); we get, since $d(a) = \alpha(a) = 0$ and $\alpha d^2 = 0$,

$$\gamma a + \alpha^2(v)\alpha d(y)[d(w), a] \in Z,$$

or

$$\gamma a - \alpha^2(v)\alpha d(y)\alpha d(w) \in Z.$$

Commuting the last equation with $d(u)$, $u \in U$, as $\gamma \in Z$, to get $\gamma[a, d(u)] = [\alpha^2(v)\alpha d(y)\alpha d(w), d(u)] = [\alpha^2(v)\alpha d(y), d(u)]\alpha d(w)$. Since γ and $[a, d(u)]$ are in Z , then $[\alpha^2(v)\alpha d(y), d(u)]\alpha d(w) \in Z$ for all $u, v \in U, w \in V$ and $y \in R$. As we have seen above that $\alpha d(V) \neq 0$, and as it is in the center Z , we conclude that $[\alpha^2(v)\alpha d(y), d(u)] \in Z$ for $u, v \in U$ and $y \in R$. In particular, $[d(w), \alpha^2(v)]\alpha d(y) + \alpha^2(v)[d(w), \alpha d(y)] \in Z$ for $v \in U, w \in V$ and $y \in R$. Thus, in view of (12) and the last equation, we conclude

that $\alpha^2(v)[d^2(w), \alpha(y)] \in Z$ for $v \in U$, $w \in V$ and $y \in R$. In particular, $\alpha^2(v)[d^2(w), \alpha(u)] \in Z$ for $u, v \in U$ and $w \in V$. In (1) replace u by $d(w)$, $w \in V$ and r by u , then $[d^2(w), \alpha(u)] \in Z$ for $w \in V$, $u \in U$. If $[d^2(w), \alpha(u)] \neq 0$, as it is in the center Z , $\alpha^2(v) \in Z$ for all $v \in U$ since R is prime. That is $[a, [a, v]] \in Z$ for all $v \in U$. By Theorem 2 $a \in Z$; a contradiction. Hence $[d^2(w), \alpha(u)] = 0$ for all $u \in U$, $w \in V$. Since $a \notin Z$, then $\alpha \neq 0$. By Theorem 2 of [2], $d^2(V) \subset Z$. Thus we have $a \in R$ is such that $d(a) = 0$, $[a, d(V)] \subset Z$ and $d^2(V) \subset Z$ where $V \not\subset Z$ is a Lie ideal of R and $d \neq 0$ is a derivation of R . So, by Theorem 1, $a \in Z$. This proves the Theorem.

An immediate consequence of Theorem 4 is the following theorem which extends Theorem 1 of [2].

Theorem 5. *Let R be a prime ring, $\text{char } R \neq 2$, and let U be a Lie ideal of R . If $d \neq 0$ is a derivation of R such that $d^2(U) \subset Z$ then $U \subset Z$.*

PROOF. Suppose on the contrary that $U \not\subset Z$. By hypothesis, $d^2(u) \in Z$ for all $u \in U$. If $u, v \in U$ then $d^2(u), d^2(v) \in Z$ and $d^2[u, v] \in Z$; that is, $[d^2(u), v] + 2[d(u), d(v)] + [u, d^2(v)] \in Z$ and so $2[d(u), d(v)] \in Z$, in consequence of which we get $[d(u), d(v)] \in Z$ for all $u, v \in U$. By Theorem 4 $d(U) \subset Z$ and so $U \subset Z$ by lemma 6 of [2], a contradiction. Hence $U \subset Z$.

Now we are in position to prove a result which generalizes simultaneously those of Theorems 2, 4 and 1 of [2].

Theorem 6. *Let R be a prime ring, $\text{char } R \neq 2$, and let $U \not\subset Z$ be a Lie ideal of R . Suppose that δ and d are derivations of R such that $\delta d(U) \subset Z$. Then either $\delta = 0$ or $d = 0$.*

PROOF. Suppose that $d \neq 0$ and $\delta \neq 0$. By hypothesis, $\delta d(u) \in Z$ for all $u \in U$. If $u \in U$, $r \in R$ then $\delta d[u, r] \in Z$ and $\delta d(u) \in Z$. Therefore, $[\delta d(u), r] + [d(u), \delta(r)] + [\delta(u), d(r)] + [u, \delta d(r)] \in Z$; that is $[d(u), \delta(r)] + [\delta(u), d(r)] + [u, \delta d(r)] \in Z$ for all $u \in U$, $r \in R$. Replace r by $d(v)$ where $v \in V$, as $d(v) \in U$ and $\delta d(U) \subset Z$, we get $[\delta(u), d^2(v)] \in Z$ for all $u \in U$, $v \in V$. By theorem 4, $d^2(V) \subset Z$ since $\delta \neq 0$. Since V is a Lie ideal of R and $d \neq 0$, then by Theorem 5 $V \subset Z$ and so $U \subset Z$, a contradiction. Hence either $d = 0$ or $\delta = 0$. This completes the proof of Theorem 6.

We are closing this paper by proving the following theorem which extends some due to Awtar [1, Theorems 1 and 2].

THEOREM 7. *Let R be a prime ring of characteristic different from 2. Let d be a nonzero derivation of R , and U a Lie ideal of R with $[u, d(u)] \in Z$ for all $u \in U$. Then $U \subset Z$.*

PROOF. By Lemma 2 of [1], $[[d(r), u], u] \in Z$ for all $u \in U$, $r \in R$. Its linearization on $u = u + d(v)$ where $v \in V$ yields on expansion $[[d(r), d(v)], u] + [[d(r), u], d(v)] \in Z$ for all $u \in U$, $v \in V$ and $r \in R$. In particular, $[d(v), [d(v), u]] \in Z$ for all $u \in U$, $v \in V$. If $U \not\subset Z$, then by Theorem 2 $d(V) \subset Z$ and so $V \subset Z$ by Lemma 6 of [2]. Thus by lemma 1 of [4] we conclude that $U \subset Z$, a contradiction. Hence $U \subset Z$.

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