

On the Lebesgue function on infinite interval, I

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1. The approximation of the Lebesgue function on a finite interval was extensively discussed by G. GRÜNWARD and P. TURÁN [1]. In this paper we extend their investigation to infinite interval.

Moreover we establish the convergence of the Lagrange interpolation polynomial on the infinite interval.

Let a weight function

$$(1.1) \quad p(x) \equiv p > 0$$

be given, such that

$$(1.2) \quad \int_{-1}^1 p(x) dx < \infty.$$

Clearly, there exists a unique orthonormal polynomial system $\{\omega_n(x)\}_{n=0}^{\infty}$ such that

$$\int_{-1}^1 \omega_n(x) \omega_m(x) p(x) dx = \delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}.$$

A special case of this system is the system of the so called Jacobi polynomials, where

$$p(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad 0 > \alpha, \beta > -1.$$

Let $\omega_n(x_{v,n})=0$, where

$$(1.3) \quad -1 < x_{n,n} < \dots < x_{v+1,n} < x_{v,n} < \dots < x_{1,n} < 1.$$

The Lagrange interpolation polynomial $L_n(x)$ for prescribed values $\{y_{v,n}\}_{n=1}^{\infty}$ such that

$$L_n(x_{v,n}) = y_{v,n} \quad (v = 1, 2, \dots, n, \quad n = 1, 2, 3, \dots)$$

can be explicitly written as

$$(1.4) \quad L_n(x) = \sum_{v=1}^n y_{v,n} l_{v,n}(x)$$

where as usual

$$l_{v,n}(x) = \frac{\omega_n(x)}{\omega'_n(x_{v,n})(x-x_{v,n})} \quad l_{v,n}(x_{j,n}) = \delta_{vj}.$$

The Lebesgue function is given as

$$(1.5) \quad \lambda_n(x) \stackrel{\text{def}}{=} \sum_{v=1}^n |l_{v,n}(x)|$$

and the Lebesgue constant is defined as

$$\lambda_n = \max_{x \in [-1, 1]} \{\lambda_n(x)\}.$$

G. GRÜNWARD and P. TURÁN [1] proved that if (1.1), is true then

$$\lambda_n(x) \leq C_1 \sqrt{n}, \quad x \in (-1, 1)$$

and

$$\lambda_n(x) \leq C_2 n, \quad x \in [-1, 1],$$

where C_1 and C_2 are suitable constants.

2. In this paper the approximation of $\lambda_n(x)$ on the interval $[0, \infty)$ will be considered.

Let a weight function $p(x)$ such that

$$(2.1) \quad \int_0^\infty x^k p(x) dx < \infty \quad (k = 0, 1, 2, \dots)$$

be given.

Clearly, there exists a unique system of orthonormal polynomials $\{\omega_n^*(x)\}_{n=0}^\infty$ such that $\omega_n^*(x)$ has exactly n simple zeros, i.e.

$$(2.2) \quad 0 < x_1 < x_2 < \dots < x_v < x_{v+1} < \dots < x_n < \infty \quad (n = 1, 2, 3, \dots).$$

It is plausible to choose our x_v -points as the zeros of n -th Laguerre polynomial defined by

$$(2.3) \quad \omega_n^*(x) = \frac{1}{n!} x^{-\alpha} e^x (e^{-x} x^{n+\alpha})^{(n)} \quad (\alpha > -1),$$

then in this case

$$p(x) = x^\alpha e^{-x}.$$

We shall now assert

Theorem 1. Let $p(x)x^{-\alpha}e^x \equiv m > 0$, $\alpha > -1$ and

$$\int_0^\infty x^k p(x) dx < \infty \quad (k = 0, 1, 2, \dots).$$

Then the Lebesgue function using the nodes (2.2) satisfies the following condition:

$$\lambda_n(x) = \begin{cases} O(1)x^{-\frac{\alpha}{2} + \frac{1}{4}} n^{\frac{1}{4}} & \left(0 < x \leq A, \alpha > -\frac{1}{2}\right), \\ O(1)n^{\frac{1}{4}} & \left(0 \leq x \leq A, -1 < \alpha \leq -\frac{1}{2}\right), \end{cases}$$

where A is an arbitrary fix real number.

Remark. If $p(x) = x^\alpha e^{-x}$, then $m = 1$.

3. A very useful role will be played by the following statement.

Lemma 1. *The Cotes numbers belonging to the system (2.2) satisfy the relations*

$$(3.1) \quad \int_0^{\infty} l_v(x) p(x) dx = \int_0^{\infty} l_v(x)^2 p(x) dx \quad (1 \leq v \leq n, n = 1, 2, 3, \dots).$$

PROOF. Let

$$I = \int_0^{\infty} l_v(x)[l_v(x)-1]p(x)dx, \quad l_v(x) = \frac{\omega_n^*(x)}{\omega_n^*(x_v)(x_v-x)}.$$

Clearly $l_v(x_j)=0$, $v \neq j$ and $l_v(x_v)=1$, $j, v=1, 2, \dots, n$, $n=1, 2, 3, \dots$. Also

$$I = \int_0^{\infty} \omega_n^*(x) q_{n-2}(x) p(x) dx,$$

where $q_{n-2}(x)$ is a polynomial of degree $n-2$. Hence from the orthogonality of $\{\omega_n^*(x)\}_{n=0}^{\infty}$, I is equal to zero and the lemma is proved.

It is also easy to show that the orthogonality of $\{\omega_n^*(x)\}_{n=0}^{\infty}$ implies the equation

$$(3.2) \quad \int_0^{\infty} l_i(x) l_j(x) p(x) dx = \begin{cases} 0, & i \neq j, \\ \mu_i, & i = j, \end{cases}$$

where from (3.1) μ_i ($i=1, 2, \dots, n$, $n=1, 2, 3, \dots$) are the Cotes numbers.

4. We shall now present the proof of Theorem 1.

Observe that

$$(4.1) \quad \sum_{v=1}^n l_v(x) \equiv 1,$$

then from Lemma 1:

$$(4.2) \quad \sum_{v=1}^n \int_0^{\infty} l_v^2(x) p(x) dx = \sum_{v=1}^n \int_0^{\infty} l_v(x) p(x) dx = \int_0^{\infty} p(x) dx.$$

This integral is independent of n .

Also, from (3.2) and (4.2)

$$(4.3) \quad \int_0^{\infty} \left\{ \sum_{v=1}^n l_v(x) \right\}^2 p(x) dx = \int_0^{\infty} \left\{ \sum_{v=1}^n l_v(x)^2 \right\} p(x) dx = \int_0^{\infty} p(x) dx < \infty.$$

Define $\varepsilon_v = \text{sign} \{l_v(x_0)\}$, where $x_0 \geq 0$ is an arbitrary fix value, and let

$$(4.4) \quad \Psi_n(x) = \sum_{v=1}^n \varepsilon_v l_v(x).$$

The polynomial $\psi_n(x)$ is of degree $n-1$ and can be expanded by the Fourier series of Laguerre orthonormal polynomials

$$(4.5) \quad \Phi_k(x) = \Gamma(\alpha+1)^{-\frac{1}{2}} \binom{n+\alpha}{n}^{-\frac{1}{2}} L_k^{(\alpha)}(x) \quad (k=0, 1, 2, \dots).$$

Thus

$$(4.6) \quad \Psi_n(x) = \sum_{k=0}^{n-1} c_k \Phi_k(x)$$

and from Cauchy's inequality

$$(4.7) \quad [\Psi_n(x)]^2 \leq \sum_{k=0}^{n-1} c_k^2 \sum_{k=0}^{n-1} \{\Phi_k(x)\}^2.$$

That is,

$$(4.8) \quad \int_0^\infty [\Psi_n(x)]^2 x^\alpha e^{-x} dx = \int_0^\infty \left\{ \sum_{k=0}^{n-1} c_k \Phi_k(x) \right\}^2 x^\alpha e^{-x} dx = \sum_{k=0}^{n-1} c_k^2 \leq \\ \leq \frac{1}{m} \int_0^\infty \left[\sum_{v=1}^n \varepsilon_v l_v(x) \right]^2 p(x) dx.$$

From (3.1) and (4.3) the last term in (4.8) gives

$$(4.9) \quad \sum_{k=0}^{n-1} c_k^2 \leq \frac{1}{m} \int_0^\infty p(x) dx < \infty.$$

This value, which is independent of n , clearly satisfies relation

$$(4.10) \quad [\Psi_n(x)]^2 = O(1) \sum_{k=0}^{n-1} [\Phi_k(x)]^2.$$

We have that (SZEGŐ [2])

$$(4.11) \quad L_n^{(\alpha)}(x) = O(1) x^{-\frac{\alpha}{2} + \frac{1}{4}} n^{\frac{\alpha}{2} - \frac{1}{4}} \left(0 \leq x \leq A, \alpha \geq -\frac{1}{2} \right)$$

and

$$(4.12) \quad L_n^{(\alpha)}(x) = O(1) n^{\frac{\alpha}{2} - \frac{1}{4}} \left(0 \leq x \leq A, -1 < \alpha \leq -\frac{1}{2} \right).$$

Relations (4.5), (4.11) and (4.12) give directly

$$(4.13) \quad \sum_{k=0}^{n-1} [\Phi_k(x)]^2 = \Gamma(\alpha+1)^{-1} \sum_{k=0}^{n-1} \binom{k+\alpha}{k}^{-1} [L_k^{(\alpha)}(x)]^2 = \\ = O(1) x^{-\alpha + \frac{1}{2}} n^{\frac{1}{2}} \left(x \in (0, A], \alpha \geq -\frac{1}{2} \right),$$

$$(4.14) \quad \sum_{k=0}^{n-1} [\Phi_k(x)]^2 = O(1) n^{\frac{1}{2}} \left(x \in [0, A], -1 < \alpha \leq -\frac{1}{2} \right).$$

Hence, from (4.4), (4.7), (4.9) and (4.10) we have

$$(4.15) \quad \begin{aligned} \Psi_n(x_0) &= \lambda_n(x_0) = \sum_{v=1}^n |l_v(x_0)| = \\ &= \begin{cases} O(1)x_0^{-\frac{\alpha}{2} + \frac{1}{4}} n^{\frac{1}{4}} & \left(x_0 \in (0, A), \alpha \cong -\frac{1}{2} \right), \\ O(1)n^{\frac{4}{1}} & \left(x_0 \in [0, A], -1 < \alpha \cong \frac{1}{2} \right). \end{cases} \end{aligned}$$

Thus, the theorem is proven, because x_0 is an arbitrary value.

5. It is interesting at this point to consider a function

$$(5.1) \quad f(x) = e^{ax} \varphi(x) \quad (a > 0, x \geq 0),$$

where $\varphi(x) \in \text{Lip}_M \gamma \left(\frac{1}{2} < \gamma \leq 1 \right)$.

Let us define polynomials $L_n(x; f)$ of degree $n-1$, satisfying the following equalities, on the roots of Laguerre polynomials

$$(5.2) \quad \begin{aligned} L_n(x_v; f) &= e^{-ax_v} f(x_v) = \varphi(x_v) = y_v \\ (v &= 1, 2, 3, \dots, n, n = 1, 2, 3, \dots). \end{aligned}$$

The polynomials $L_n(x; f)$ have the explicit forms

$$(5.3) \quad L_n(x; f) = \sum_{v=1}^n y_v l_v(x),$$

where

$$(5.4) \quad l_v(x) = \frac{L_n^{(a)}(x)}{L_n^{(a)'}(x_v)(x - x_v)}.$$

We shall now prove the following statement.

Theorem 2. *If $f(x)$ satisfies the condition (5.1). Then*

$$|f(x) - e^{ax} L_n(x; f)| = \begin{cases} O(1)x^{\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{1}{4} - \frac{\gamma}{2}} & \left(x \in (0, A], \alpha \cong -\frac{1}{2} \right), \\ O(1)n^{\frac{1}{4} - \frac{\gamma}{2}} & \left(x \in [0, A], -1 < \alpha \cong -\frac{1}{2} \right), \end{cases}$$

where A is an arbitrary fix real number.

PROOF. Let $x_v = x_v(\alpha)$ denote the roots of polynomials $L_n^{(a)}(x)$, then the following inequalities are true (SZEGŐ [2])

$$(5.5) \quad \begin{aligned} c_1 \frac{v^2}{2n + \alpha + 1} &< x_v < c_2 \frac{v^v}{2n + \alpha + 1} \\ (v &= 1, 2, \dots, n, n = 1, 2, 3, \dots), \end{aligned}$$

where c_1 and c_2 constants are independent of v and n .

Let $\varphi(x) \in \text{Lip}_M \gamma$, $\frac{1}{2} < \gamma \leq 1$, $x \geq 0$ and $x = \frac{x_n}{2}(u+1)$, $x_n = x_n(\alpha)$ then

$$(5.6) \quad g(u) \stackrel{\text{def}}{=} \varphi\left(\frac{x_n}{2}(u+1)\right) = \varphi(x).$$

Obviously, if $x \in [0, x_n]$ then $u \in [-1, 1]$ and since $|u' - u''| \leq \delta$, $u', u'' \in [-1, 1]$ we have

$$(5.7) \quad \omega(\delta; g) = \omega\left(\frac{x_n}{2}\delta; \varphi\right) = O(1)\left(\frac{x_n}{2}\delta\right)^\gamma,$$

where $\omega(\cdot; g)$ and $\omega(\cdot; \varphi)$ denote the moduli of continuity of g and φ , respectively.

If $u \in [-1, 1]$, $u = \cos \vartheta$, then $\Psi(\vartheta) \stackrel{\text{def}}{=} g(\cos \vartheta)$ is an even function, so its Jackson-mean [3] $J_m(\vartheta, \Psi)$ is a pure cosine polynomial of order $2m-2$, therefore

$$(5.8) \quad J_m(\arccos u, \Psi) = Q_{2m-2}(u; g)$$

is an algebraic polynomial of degree $2m-2$ and

$$(5.9) \quad Q_{2m-2}(u, g) = \frac{3}{\pi m(2m^2+1)} \int_0^{\pi/2} \{g[\cos(\arccos u + 2t)] + g[\cos(\arccos u - 2t)]\} \left(\frac{\sin mt}{\sin t}\right)^4 dt.$$

It is well known that

$$(5.10) \quad \frac{6}{\pi m(2m^2+1)} \int_0^{\pi/2} \left(\frac{\sin mt}{\sin t}\right)^4 dt = 1.$$

By using relations (5.9), (5.10) and (5.7) it is easy to verify that

$$(5.11) \quad |Q_{2m-2}(u, g) - g(u)| = O(1) \left\{ \left(\frac{x_n |u|}{m^2}\right)^\gamma + \left(\frac{x_n \sqrt{1-u^2}}{m}\right)^\gamma \right\}.$$

From (5.11) and (5.6) if $u = \frac{2x-x_n}{x_n}$ and $x \in [0, x_n]$ we have

$$(5.12) \quad \begin{aligned} \left| Q_{2m-2}\left(\frac{2x-x_n}{x_n}; \varphi\right) - \varphi(x) \right| &= O(1) \left\{ \left(\frac{|2x-x_n|}{m^2}\right)^\gamma + \left(\frac{\sqrt{x_n x - x^2}}{m}\right)^\gamma \right\} = \\ &= O(1) \left\{ \left(\frac{x_n}{m^2}\right)^\gamma + \left(\frac{\sqrt{x_n x}}{m}\right)^\gamma \right\}, \end{aligned}$$

where equality

$$Q_{2m-2}^*(x; \varphi) \equiv Q_{2m-2}\left(\frac{2x-x_n}{x_n}; \varphi\right)$$

implies that it is a polynomial of degree $2m-2$.

If $m = \left\lfloor \frac{n}{2} \right\rfloor$, then $2m-2 \leq n-1$ and in this case we have

$$(5.13) \quad Q_{2m-2}^*(x; \varphi) \equiv \sum_{v=1}^n Q_{2m-2}(x_v; \varphi) l_v(x).$$

From (5.12), (5.13) and (4.12) we have, that for $x \in (0, A]$, $\alpha \geq -\frac{1}{2}$, $m = \left\lfloor \frac{n}{2} \right\rfloor$, $x_n = x_n(\alpha) = O(1)n$

$$(5.14) \quad \begin{aligned} |\varphi(x) - L_n(x; \varphi)| &\leq |\varphi(x) - Q_{2m}^*(x; \varphi)| + \sum_{v=1}^n |Q_{2m-2}^*(x_v) - \varphi(x_v)| |l_v(x)| \leq \\ &\leq O(1)n^{-\frac{\gamma}{2}} + O(1)n^{-\frac{\gamma}{2}} \sum_{0 < x_v < 2A} |l_v(x)| + \\ &+ O(1)n^{-\frac{\gamma}{2}} \left\{ \sum_{x_v > 2A} |L_n^{(\alpha)}(x)| |L_n^{(\alpha)'}(x_v)|^{-1} \right\}. \end{aligned}$$

The following inequality is true, if $1 \leq m \leq 2n-1$ (SZEGŐ [2])

$$(5.15) \quad \sum_{v=1}^n x_v^{\frac{m}{2} - \frac{1}{2}} |L_n^{(\alpha)'}(x_v)|^{-1} \leq \sqrt{n} \left\{ \sum_{v=1}^n x_v^{m-1} [L_n^{(\alpha)'}(x_v)]^{-2} \right\}^{\frac{1}{2}} = O(1)n^{\frac{1}{2} - \frac{\alpha}{2}}.$$

Using (5.1), Theorem 1, (5.14), (4.12) and (5.15) we get

$$\begin{aligned} |f(x) - e^{\alpha x} L_n(x; f)| &= e^{\alpha x} |\varphi(x) - L_n(x; \varphi)| = \\ &= \begin{cases} O(1)x^{\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{1}{4} - \frac{\gamma}{2}}, & \text{for } \alpha \geq -\frac{1}{2}, x \in (0, A], \\ O(1)n^{\frac{1}{4} - \frac{\gamma}{2}}, & \text{for } -1 < \alpha \leq -\frac{1}{2}, x \in [0, A]. \end{cases} \end{aligned}$$

This completes the proof of Theorem 2.

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References

- [1] G. GRÜNWARD—P. TURÁN, Über Interpolation, *Annali della Scuola Sup. de Pisa*, (2) 7 (1938), 137—146.
- [2] G. SZEGŐ, Orthogonal polynomials. *Amer. Math. Soc. Coll. Publ.*, XXIII, 1959.
- [3] D. JACKSON, The theory of approximation, *Amer. Math. Soc. Coll. Publ.*, 11 (1930).

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