Rees algebras and their varieties

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Following [3, 8], a subalgebra $\mathfrak{B}$ of an algebra $\mathfrak{A}$ is called a *Rees subalgebra* whenever there exists a congruence θ on $\mathfrak{A}$ such that $\langle x, y\rangle\in\theta$ if and only if either $x=y$ or both x, y are elements of $\mathfrak{B}$. In this case, θ is called a *Rees congruence* on $\mathfrak{A}$ induced by $\mathfrak{B}$. Rees congruences were introduced by D. REES [6] for semigroups; Rees congruences on lattices were used by G. SZÁSZ [7] for a construction of special kind.

The present paper deals with algebras having Rees subalgebras only. They are called *Rees algebras*. Rees algebras are closely related to *Hamiltonian algebras,* see [5], and thus also with algebras having the *Congruence Extension Property,* see [4]. For basic concepts and notations used in this paper see [2]. We will write A, B, etc. for the universes of the algebras $\mathfrak{A}$, $\mathfrak{B}$, etc.

1. Basic concepts

In the sense of [5], an algebra $\mathfrak{A}$ is called *Hamiltonian* if and only if every subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is a block of the congruence $\theta(B\times B)$, where $\theta(B\times B)$ denotes the smallest congruence on $\mathfrak{A}$ collapsing B. In some particular cases, this congruence may be of the form $\theta(B\times B)=B\times B\cup\omega_A$, where ω_A denotes the diagonal of A. However, this form of $\theta(B\times B)$ is identical to that of [3]. In this way we introduce

Definition 1. An algebra $\mathfrak{A}$ is called a *Rees algebra* if $B\times B\cup\omega_A$ is a congruence on $\mathfrak{A}$ for every subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. A variety $\mathscr{V}$ is a *Rees variety* if each $\mathfrak{A}\in\mathscr{V}$ is a Rees algebra.

Thus a Rees algebra is a special case of a Hamiltonian algebra. As it was proved by E. W. KISS [4], every variety of Hamiltonian algebras has the Congruence Extension Property (briefly CEP). For this reason recall

Definition 2. An algebra A has CEP if every congruence on an arbitrary subalgebra of $\mathfrak{A}$ is a restriction of some congruence on $\mathfrak{A}$.

For Rees algebras, it will be useful to modify the definition of CEP in the following sense:

Definition 3. An algebra $\mathfrak{A}$ satisfies *strong CEP* whenever $\theta\cup\omega_A\in\mathrm{Con}\,\mathfrak{A}$ for any congruence θ on a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.

A class of algebras $\mathscr{C}$ has *CEP (strong CEP)* if each algebra in $\mathscr{C}$ has CEP (strong CEP).

Apparently, strong CEP implies CEP and any algebra having strong CEP is a Rees one. A natural question arises: does there exist a Rees algebra which has not strong CEP? The answer is positive:

Example 1. Let $\mathfrak{A}$ be a four element groupoid given by its multiplicative table:

$\cdot$	a	b	c	z
a	c	c	a	a
b	c	c	a	c
c	c	c	b	b
z	a	c	b	z

Clearly a subset $\mathfrak{B}=\{a, b, c\}$ forms a subgroupoid of $\mathfrak{A}$ and $\mathfrak{A}$ has no other proper subgroupoid. Clearly $\mathfrak{A}$ is a Rees groupoid since $B\times B\cup\omega_A$ is a congruence on $\mathfrak{A}$. However, $\mathfrak{A}$ has no strong CEP since the equivalence θ with classes $\{a, b\}$, $\{c\}$ is a congruence on $\mathfrak{B}$ although $\theta\cup\omega_A$ is not a congruence on $\mathfrak{A}$ because

$$\langle a, b\rangle\in\theta, \quad \langle z, z\rangle\in\omega_A \quad \text{but} \quad \langle a, c\rangle = \langle a\cdot z, b\cdot z\rangle\notin\theta\cup\omega_A.$$

2. Rees algebras

Theorem 1. *For an algebra $\mathfrak{A}$, the following conditions are equivalent:*

(1) *$\mathfrak{A}$ is a Rees algebra;*

(2) *every subalgebra of $\mathfrak{A}$ generated by two elements is Rees;*

(3) *for every unary algebraic function φ over $\mathfrak{A}$ and any two elements a, b of $\mathfrak{A}$ we have either* (i) *$\varphi(a)=\varphi(b)$, or* (ii) *$\varphi(a)=s(a, b)$, $\varphi(b)=t(a, b)$ for some binary polynomials s and t of $\mathfrak{A}$.*

PROOF. (1)$\Rightarrow$(2) is trivial. Prove (2)$\Rightarrow$(3): Let φ be a unary algebraic function over $\mathfrak{A}$ and $a, b\in\mathfrak{A}$. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ generated by elements a, b. Then, using (2), $B\times B\cup\omega_A$ is a congruence on $\mathfrak{A}$ containing the pair $\langle a, b\rangle$. Consequently, also $\langle\varphi(a), \varphi(b)\rangle\in B\times B\cup\omega_A$, whence (3) is evident.

(3)$\Rightarrow$(1): Let $\mathfrak{B}$ be an arbitrary subalgebra of $\mathfrak{A}$. We have to show that the binary relation $B\times B\cup\omega_A$ is a congruence on A. Evidently, it suffices to verify the Substitution Property only: choose $\langle a, b\rangle\in B\times B\cup\omega_A$ and a unary algebraic function φ over $\mathfrak{A}$. Applying the hypothesis (3), we have either (i) $\varphi(a)=\varphi(b)$ or (ii) $\varphi(a)=s(a, b)$ and $\varphi(b)=t(a, b)$ for some binary polynomials s, t of $\mathfrak{A}$. Clearly the first case means that $\langle\varphi(a), \varphi(b)\rangle\in\omega_A$. Suppose $\varphi(a)\neq\varphi(b)$. Then $\langle a, b\rangle\in B\times B$ and thus by (ii),

$$\langle\varphi(a), \varphi(b)\rangle = \langle s, (a, b), t(a, b)\rangle\in B\times B.$$

Summarizing, we get

$$\langle\varphi(a), \varphi(b)\rangle\in B\times B\cup\omega_A$$

which clearly implies the Substitution Property.

Example 2. (1) For a group $\mathfrak{G}$, the following two conditions are equivalent:
(a) $\mathfrak{G}$ is a Rees group;
(b) $\mathfrak{G} \cong Z_p$, the cyclic group of prime order.

PROOF. (a)$\Rightarrow$(b): Let $\mathfrak{H}$ denote a subgroup of $\mathfrak{G}$. By the hypothesis, $H \times H \cup \omega_A$ is a congruence on $\mathfrak{G}$. Now, the regularity of groups (see e.g. [1]) implies $\mathfrak{H} = \mathfrak{G}$ or $\mathfrak{H} = \{0\}$, whence the conclusion $\mathfrak{G} \cong Z_p$, p prime, follows. The converse implication (b)$\Rightarrow$(a) is trivial since Z_p, p prime, has no proper subgroup.

(2) For a semilattice $\mathfrak{S}$, the following two conditions are equivalent:
(a) $\mathfrak{S}$ is a Rees semilattice;
(b) the length of $\mathfrak{S}$ is at most 1.

PROOF. (b)$\Rightarrow$(a) is trivial. Prove (a)$\Rightarrow$(b). Suppose that the length of $\mathfrak{S}$ is at least 2. Then $\mathfrak{S}$ contains a three element chain $a<b<c$. Consequently, $\{a, c\}$ is a subsemilattice of $\mathfrak{S}$ and, by the hypothesis, $\{a, c\}$ is a block of some congruence on $\mathfrak{S}$. However, any congruence block is convex, which is a contradiction.

(3)$\Rightarrow$(2) holds also for lattices.

(4) Every unary algebra is a Rees algebra.

PROOF. This is a trivial consequence of Theorem 1 (3).

Proposition 1. *Being a Rees algebra is hereditary for subalgebras and homomorphic images.*

PROOF. The first assertion is evident. Further, let $h: \mathfrak{A} \to \mathfrak{A}'$ be a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$ and $\mathfrak{B}'$ be a subalgebra of $\mathfrak{A}'$. Put $\mathfrak{B} = h^{-1}(\mathfrak{B}')$. Then, by the hypothesis, $B \times B \cup \omega_A$ is a congruence on $\mathfrak{A}$. It is routine to verify the formula

$$(h \times h)(B \times B \cup \omega_A) = B' \times B' \cup \omega_{A'}$$

which implies that $B' \times B' \cup \omega_{A'}$ is a subalgebra of the square $\mathfrak{A}' \times \mathfrak{A}'$. Thus $B' \times B' \cup \omega_{A'}$ is a congruence on $\mathfrak{A}'$ which finishes the proof.

Remark 1. The class of all Rees algebras of the same type is not closed under forming direct products: consider the two-element semilattice (or lattice) $\mathfrak{C}_2$. Then the direct product $\mathfrak{C}_2 \times \mathfrak{C}_2$ is not a Rees semilattice (or lattice, respectively).

3. Characterizations of Rees varieties

Although Rees algebras of the same type need not form a variety, there exist varieties of algebras whose all members are Ress algebras as we can see in Example 2 (4). It motivates our aim to characterize such varieties.

Let $\mathscr{V}$ be a variety. An n-ary polynomial p of $\mathscr{V}$ is called *essentially k-ary* $(0 \leqq k \leqq n)$ on the variety $\mathscr{V}$, if the polynomial p on the countably generated free algebra of $\mathscr{V}$ depends on exactly k variables, see [1]. We say that $\mathscr{V}$ *is at most unary* if every polynomial p of $\mathscr{V}$ is either essentialy unary or essentially nullary.

Further, denote by $F_n(x_1, \dots, x_n)$ the free algebra of $\mathscr{V}$ generated by the set of free generators $\{x_1, \dots, x_n\}$.

Recently, E. W. Kiss [4] has proved that any variety of Hamiltonian algebras has CEP. In this section we discuss the relationship between Rees varieties and strong CEP; moreover, we present different characterizations of Rees varieties.

Theorem 2. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) *$\mathcal{V}$ is a Rees variety;*
(2) *$\mathcal{V}$ is at most unary;*
(3) *every algebraic function over $\mathfrak{A}\in\mathcal{V}$ is either a constant or a polynomial;*
(4) *for every unary algebraic function φ over $\mathfrak{A}\in\mathcal{V}$ and any two elements a, b of $\mathfrak{A}$, either* (i) $\varphi(a)=\varphi(b)$ *or* (ii) *there exists a unary polynomial u of $\mathcal{V}$ such that* $\varphi(a)=u(a)$, $\varphi(b)=u(b)$;
(5) *$\mathcal{V}$ has strong* CEP.

Proof. (1)$\Rightarrow$(2): Let $\mathcal{V}$ be a Rees variety. Consider the free algebra $F_{2+n}(x, y, z_1, \ldots, z_n)$. Suppose f is an $(n+1)$-ary polynomial of $\mathcal{V}$ depending on the first variable. Then $\varphi(v)=f(v, z_1, \ldots, z_n)$ is a unary algebraic function over $F_{2+n}(x, y, z_1, \ldots, z_n)$ and, by Theorem 1 (3), we have either (i) $f(x, z_1, \ldots, z_n)=$ $=f(y, z_1, \ldots, z_n)$ or (ii) $f(x, z_1, \ldots, z_n)=s(x, y)$ and $f(y, z_1, \ldots, z_n)=t(x, y)$ for some binary polynomials s and t of $\mathcal{V}$. The case (i) is impossible since f depends on the first variable. The second case (ii) implies $f(x, z_1, \ldots, z_n)=t(x, x)$, i.e. f is at most unary.

The implications (2)$\Rightarrow$(3)$\Rightarrow$(4) are evident. Prove (4)$\Rightarrow$(5). Let $\mathfrak{B}$ be an arbitrary subalgebra of an algebra $\mathfrak{A}\in\mathcal{V}$. Further, let θ be a congruence on $\mathfrak{B}$. We have to show that the trivial extension $\theta\cup\omega_A$ of θ is a congruence on $\mathfrak{A}$. Let φ be a unary algebraic function over $\mathfrak{A}$ and $\langle a, b\rangle\in\theta\cup\omega_A$. Suppose $\varphi(a)\neq\varphi(b)$. Then $\langle a, b\rangle\in\theta$ and, by the hypothesis (4),

$$\varphi(a) = u(a), \quad \varphi(b) = u(b)$$

hold for some unary polynomial u of $\mathcal{V}$. Thus

$$\langle\varphi(a), \varphi(b)\rangle = \langle u(a), u(b)\rangle\in\theta$$

which completes the proof.

The implication (5)$\Rightarrow$(1) is trivial.

Remark 2. Examples 2 (1), (2) and (3) show that condition (4) of Theorem 2 is weaker than part (3) of Theorem 1, i.e. a Rees algebra alone need not be unary. Further, we have proved that a Rees variety is equivalent to a variety having strong CEP although this need not be true for a single algebra, see Example 1. The following propositions point out which single algebras has strong CEP.

Proposition 2. *If $\mathfrak{A}\times\mathfrak{A}$ is a Rees algebra then $\mathfrak{A}$ has strong* CEP *(and thus* CEP*).*

Proof. Let θ be an arbitrary congruence on a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Evidently, θ is a subalgebra of the square $\mathfrak{B}\times\mathfrak{B}$ and so it is a subalgebra of the Rees algebra $\mathfrak{A}\times\mathfrak{A}$. The diagonal ω_A has the same property and, moreover, $\theta\cap\omega_A\neq\emptyset$. Then, using the former result of [8; Proposition 2.1, p. 230], $\theta\cup\omega_A$ is also a subalgebra of $\mathfrak{A}\times\mathfrak{A}$ which completes the proof.

Proposition 3. *Any idempotent Rees algebra has strong CEP.*

Proof. Let $\mathfrak{B}$ be a subalgebra of an idempotent Rees algebra $\mathfrak{A}$. We have to hsow that $\theta\cup\omega_A\in\mathrm{Con}\,\mathfrak{A}$ for any congruence θ on $\mathfrak{B}$. Take a unary algebraic function φ over $\mathfrak{A}$ and suppose $\langle a, b\rangle\in\theta\cup\omega_A$. If $\varphi(a)\neq\varphi(b)$ then $\langle a, b\rangle\in\theta$ and, by Theorem 1 (3),

$$\varphi(a)=s(a, b), \quad \varphi(b)=t(a, b)$$

for some binary polynomials s, t of $\mathfrak{A}$. Since $\mathfrak{A}$ is idempotent, we have

$$\langle s(a, b), a\rangle=\langle s(a, b), s(a, a)\rangle\in\theta,$$

$$\langle t(a, b), a\rangle=\langle t(a, b), t(a, a)\rangle\in\theta$$

and thus

$$\langle\varphi(a), \varphi(b)\rangle=\langle s(a, b), t(a, b)\rangle\in\theta.$$

Hence $\langle\varphi(a), \varphi(b)\rangle\in\theta\cup\omega_A$, which was to be proved.

The next theorem characterizes Rees varieties in terms of subalgebras:

Theorem 3. *For a variety $\mathscr{V}$, the following three conditions are equivalent:*

(1) *$\mathscr{V}$ is a Rees variety;*

(2) *subalgebras of each $\mathfrak{A}\in\mathscr{V}$ are closed under set union;*

(3) $F_n(x_1, \ldots, x_n)=\bigcup\limits_{i=1}^{n} F_1(x_i)$.

Proof. By Theorem 2 (2), $\mathscr{V}$ is at most unary, thus evidently (1)$\Rightarrow$(2). The implication (2)$\Rightarrow$(3) is trivial. The condition (3) implies that $\mathscr{V}$ is at most unary. Thus, by Theorem 2, also (3)$\Rightarrow$(1) is true.

Remark 3. Example 2 (2) shows that the characterization (2) of Theorem 3 does not hold for a single algebra. Nevertheless, the following local version of Theorem 3 (2) can easily be verified:

Proposition 4. *If subalgebras of $\mathfrak{A}\times\mathfrak{A}$ are closed under set union then $\mathfrak{A}$ is a Rees algebra.*

Proof. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}\times\mathfrak{B}$ as well as ω_A are subalgebras of $\mathfrak{A}\times\mathfrak{A}$. By the hypothesis, also $B\times B\cup\omega_A$ is a subalgebra of $\mathfrak{A}\times\mathfrak{A}$, i.e. $B\times\times B\cup\omega_A$ is a congruence on $\mathfrak{A}$ which was to be proved.

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