

A note on strongly regular near-rings

By Y. V. REDDY and C. V. L. N. MURTY (Nagarjunanagar)

Introduction

Throughout this note by a near-ring we mean a zerosymmetric right near-ring. A near-ring N is said to be *strongly regular* if $a \in Na^2$ for each $a \in N$. In this note we prove that if N is a strongly regular near-ring, then N has maximum condition on ideals if and only if N has minimum condition on ideals. We also present an affirmative answer to an open problem raised by MASON (See remark on page 33 of [5]).

If S is a nonempty subset of a near-ring N , we write $l(S) = \{x \in N / xS = \{0\}\}$ and $r(S) = \{x \in N / Sx = \{0\}\}$. It is easy to verify that if N is a near-ring without nonzero nilpotent elements then $l(S) = r(S)$ and this is an ideal of N . For terminology and notation, we refer to [7]. Recall that a near-ring N is called regular if for each $a \in N$ there exists $x \in N$ such that $a = axa$. A near-ring N is said to have the IFP if for any a, b in N , $ab = 0$ implies $anb = 0$ for each $n \in N$.

1

For convenience, we first state a lemma whose proof follows from propositions 9.37 and 9.43 of [7].

Lemma 1. *Suppose N is a near-ring without nonzero nilpotent elements. Then*

- (i) N has the IFP,
- (ii) For any $a, b \in N$, $ab = 0$ implies $ba = 0$,
- (iii) $en = ene$ for any idempotent e and any $n \in N$.

The following lemma has been proved in several papers (eg [8]). The converse is also true, indeed in a more general set-up (see theorem 12).

Lemma 2. *Suppose N is a strongly regular near-ring. Then*

- (i) N has no nonzero nilpotent elements,
- (ii) N is regular.

In ring theory it is well known that [3, proposition 2.28] if R is a strongly regular ring then every one sided ideal is two sided. But in case of near-rings MASON [5]

showed that if N is a strongly regular near-ring with identity, then every N -subgroup of N is an ideal. Our next result is a generalization.

Lemma 3. *If N is a strongly regular near-ring, then $l(r(Na))=Na$ for each a in N . Moreover, each N -subgroup is an ideal of N .*

PROOF. Let $a \in N$. Since N is regular, there is an idempotent e in N such that $Na = Ne$. Clearly $Na \subseteq l(r(Na))$. If $x \in l(r(Na))$ then $xr(Na) = \{0\}$. Since e is an idempotent, by lemma 1 (i), we have $x - xe \in l(Ne) = r(Ne) = r(Na)$. Hence $x(x - xe) = 0$. So $xe(x - xe) = 0$, by lemma 1 (i) and hence $(x - xe)^2 = 0$. Therefore $x = xe \in Ne = Na$. Thus $Na = l(r(Na))$ is an ideal of N . It is now easy to show that each N -subgroup is an ideal of N .

Lemma 4. *If N is a strongly regular near-ring, then N is isomorphic to a subdirect product of near-rings N_i , where in each N_i , $N_i x = N_i$ for each nonzero x in N_i and each nonzero idempotent in N_i is a right identity.*

PROOF. From [2, theorem 3.2] and lemma 3, it follows that the intersection of all maximal ideals is $\{0\}$. Hence N is isomorphic to a subdirect product of simple near-rings N_i . But from lemma 3, for each $0 \neq x \in N_i$, $N_i x$ is a nonzero ideal and hence $N_i x = N_i$. Thus each N_i has no zero divisors and hence each nonzero idempotent in N_i is a right identity.

Corollary 5. ([8, theorem 2]). *If in a strongly regular near-ring N , every nonzero homomorphic image contains a nonzero distributive element then N is isomorphic to a subdirect product of near-fields and hence $(N, +)$ is abelian.*

The proof of this corollary follows directly from lemma 4 and [7, theorem 8.3, p. 237].

Corollary 6. *Let N be a strongly regular near-ring. If N contains a left identity, then $(N, +)$ is abelian.*

Corollary 7. ([4, theorem 3]). *If in a near-ring N , for each x there exists $n(x) > 1$ such that $x^{n(x)} = x$, then N is isomorphic to a subdirect product of near-rings N_i where N_i is either a near-field or $N_i x = N_i$ for each $0 \neq x$ in N_i and each nonzero idempotent element of N_i is a right identity.*

In ring theory it is well known that [3, proposition 1.29] if R is a regular ring then for any a, b in R there is an element c in R such that $Ra + Rb = Rc$. We now obtain this result for strongly regular near-rings. But it is not known whether this result is true for regular near-rings.

Lemma 8. *If N is a strongly regular near-ring then for any a, b in N there is an element c such that $Na + Nb = Nc$.*

PROOF. Since N is regular, there exists idempotents e and f such that $Na = Ne$ and $Nb = Nf$. Now Ne is an ideal by lemma 3. So, $f = f^2 = ff - f(f - fe) + f(f - fe) \in Ne + N(f - fe)$. Thus $Ne + Nf \subseteq Ne + N(f - fe)$. Clearly $f - fe \in Nf + Ne = Ne + Nf$. Hence $N(f - fe) \subseteq Ne + Nf$ and $Ne + N(f - fe) \subseteq Ne + Nf$. Therefore $Ne + Nf = Ne + N(f - fe)$. Let g be an idempotent such that $N(f - fe) = Ng$. Clearly $ge = 0$. By lemma 1 (ii), it follows that $eg = 0$ and so by lemma 1 (iii) $e(e + g) = e(e + g)e =$

$=e^3=e$ and $g(e+g)=g(e+g)g=g^3=g$. Thus $e, g \in N(e+g)$ and so, $Ne+Ng \subseteq N(e+g)$. Since $e+g \in Ne+Ng$, we have $N(e+g) \subseteq Ne+Ng$. Therefore $Ne+Ng=N(e+g)$ and $Na+Nb=Ne+Ng=N(e+g)$.

For a ring with an identity, it is well known that the minimum condition on right ideals implies the maximum condition on right ideals. Since in a strongly regular ring every one sided ideal is two sided ideal ([3, proposition 2.28]), it follows that in a strongly regular ring with identity the minimum condition on ideals implies the maximum condition on ideals. Theorems 3.2 and 2.7 of [6] shows that if N is a regular near-ring with *dcc* on N -subgroups, then it has maximum condition on N -subgroups. From lemma 3, it evidently follows that for a strongly regular near-ring the maximum condition on ideals implies the maximum condition on ideals. We now prove the converse of this result.

Theorem 9. *Suppose N is a strongly regular near-ring. If N has maximum condition on ideals, then it has minimum condition on ideals.*

PROOF. By lemma 3, for any a in N the principal ideal generated by a is Na . Since N has maximum condition on ideals, every ideal L of N is finitely generated. By lemma 8 L is a principal ideal and hence an annihilator ideal by lemma 3. Thus every ideal of N is an annihilator ideal.

Suppose $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$ is a chain of ideals. Then $l(I_1) \subseteq l(I_2) \subseteq \dots \subseteq l(I_n) \subseteq \dots$ is an ascending sequence of ideals and hence by the maximum condition there is a positive integer k such that $l(I_n) = l(I_k)$ for all $n \geq k$. Hence $r(l(I_n)) = r(l(I_k))$ for all $n \geq k$ and since every ideal is an annihilator ideal, it follows that $I_n = I_k$ for all $n \geq k$. Thus N has minimum condition on ideals.

2

In ring theory, it is well known that if every nilpotent element is in the centre then every idempotent is in the centre. Hence a regular ring without nilpotent elements is strongly regular. In [5], Mason raised the following question: If in a regular near-ring N each nilpotent element is in the centre, then is it true that N is strongly regular? In the following we prove an affirmative answer to this question. We now establish a generalized result. To this end we introduce the following

Definition 10. A near-ring N is said to be symmetric if $a^2=0$ implies $\langle a \rangle^2=\{0\}$, where $\langle a \rangle$ is the principal ideal generated by a .

Remark 11. The class of all symmetric near-rings contains the class of all near-rings without nilpotent elements, the class of all near-rings which have IFP and the class of all near-rings in which nilpotent elements are in the centre.

Theorem 12. *Suppose N is a regular near-ring which is also symmetric. Then N is a strongly regular near-ring.*

PROOF. We first show that N has no nonzero nilpotent elements. Let $s \in N$ with $s^2=0$. Then $\langle s \rangle^2=\{0\}$. If $s \neq 0$, then by regularity there exists a nonzero idempotent e in $\langle s \rangle$. Therefore $e=e^2 \in \langle s \rangle^2=\{0\}$ and this is a contradiction. Thus N has no nonzero nilpotent elements. If $a \in N$ then $a=axa$ for some $x \in N$. Since

aa is an idempotent, by lemma 1 (iii) it follows that $ax = (axa)x = a(xa)x = x(xa)x(xa) \in Na$ and hence $a = axa \in Na^2$. Thus N is strongly regular.

By remark 11 and theorem 12 we have the following

Corollary 13. *Suppose N is a regular near-ring in which every nilpotent element is in the centre. Then N is strongly regular.*

We close this article with the following

Example 14. Let $N = \{0, 1, 2, 3, 4\}$. Define addition as modulo 5 and multiplication as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	4	1	4
2	0	2	3	2	3
3	0	3	2	3	2
4	0	4	1	4	1

Then $(N, +, \cdot)$ is a near-ring (see CLAY [1]). Clearly N is a regular near-ring with no nonzero nilpotent elements. But no nonzero idempotent is in the centre of N .

Acknowledgement. The second author wishes to express thanks to the Council of Scientific and Industrial Research, New Delhi for giving financial assistance.

References

- [1] J. R. CLAY, The near-rings on groups of low order, *Math. Z.* **104** (1968), 364—371.
- [2] M. J. JOHNSON, Radicals of regular near-rings, *Monatsh. Math.* **80** (1975), 313—341.
- [3] K. KOH, Quasisimple modules and other topics in Ring theory Lectures on Rings and Modules, Springer-Verlag, New York 246 (1972), p. 323—428.
- [4] S. LIGH, On the commutativity of near-rings II, *Kyungpook Math. J.* **11** (1971), 159—163.
- [5] G. MASON, Strongly regular near-rings, *Proc. Edin. Math. Soc.* **23** (1980), 27—35.
- [6] A. OSWALD, Some topics in the structure theory of near-rings, *Doctoral Dissertation, Univ. of York* 1973.
- [7] G. PILZ, Near-rings, *North-Holland, Amsterdam* (1977).
- [8] G. SZETO, On regular near-rings with no nonzero nilpotent elements, *Math. Japon.* **79** (1974), 65—70.

DEPARTMENT OF MATHEMATICS
NAGARJUNA UNIVERSITY
NAGARJUNANAGAR—522 510
A. P. INDIA

(Received June 29, 1983.)