

U -lifters of single morphisms in the category of F -algebras

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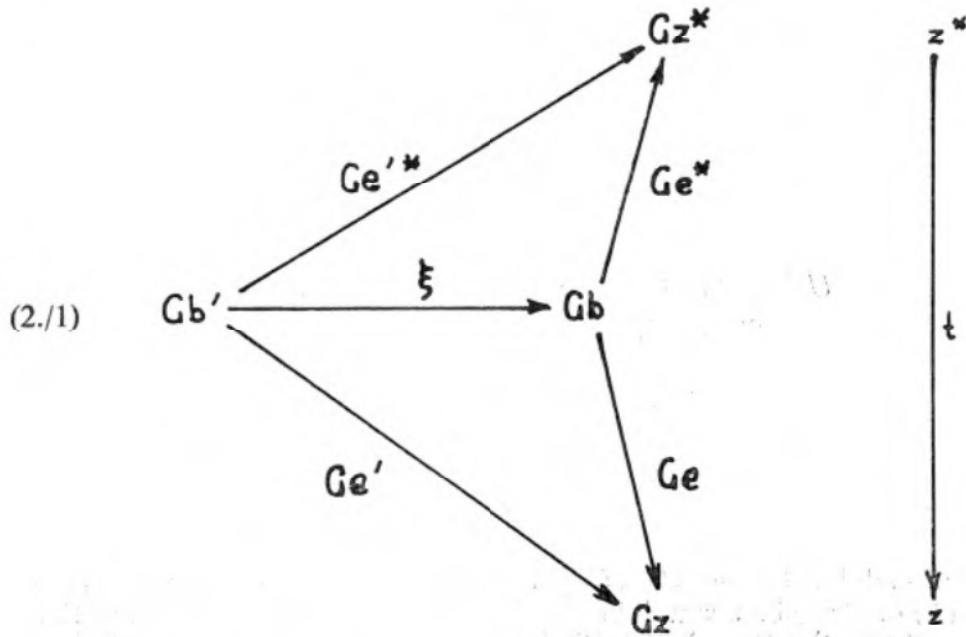
1. Introduction

Given an endofunctor $F: \mathcal{A} \rightarrow \mathcal{A}$ on the category $\mathcal{A}$ the construction of the category $\mathcal{A}(F)$ of F -algebras is well known (see e.g. in [1—5, 7, 11, 14—15]). For a morphism $\xi: U\langle a', h' \rangle \rightarrow U\langle a, h \rangle$ ($\langle a', h' \rangle, \langle a, h \rangle$ are F -algebras and $U: \mathcal{A}(F) \rightarrow \mathcal{A}$ is the canonical forgetful functor) one can define the U -lifter of ξ as in [10]. If U is presemiotopological then the existence of this U -lifter is clear since presemiotopologicality was defined by requiring U -lifters for all $U \circ D \rightarrow U\langle a, h \rangle$ cones, where D ranges on functors into $\mathcal{A}(F)$ (cf. [9—10]). Originally the more general notion of a G -lifter ($G: \mathcal{B} \rightarrow \mathcal{A}$ is a functor) for a single morphism first appeared in [8] under the name $(1_{\mathcal{B}}, G)$ -quotient.

The main aim of the present paper is to give sufficient conditions for the existence of certain U -lifters. The first result (2.1) concerns the general case of G -lifters. In (2.4) an image factorization system $(\mathcal{E}, \mathcal{M})$ will be used in order to obtain U -lifters of morphisms $\xi \in \mathcal{E}$. Finally the condition in (2.6) that F preserves certain colimits will give the existence of all U -lifters (of single morphisms).

2. Constructions providing lifters

Let $G: \mathcal{B} \rightarrow \mathcal{A}$ be a functor and $\xi: Gb' \rightarrow Gb$ be a morphism in $\mathcal{A}$; then we say that ξ has a G -lifter if there is a triple $\langle e'^*, z^*, e^* \rangle$ with $z^* \in |\mathcal{B}|$, $e'^*: b' \rightarrow z^*$, $e^*: b \rightarrow z^*$ and $(Ge^*) \circ \xi = Ge'^*$ such that for each triple $\langle e', z, e \rangle$ with $z \in |\mathcal{B}|$, $e': b' \rightarrow z$, $e: b \rightarrow z$ and $(Ge) \circ \xi = Ge'$ there exists a unique morphism $t: z^* \rightarrow z$ with $t \circ e^* = e$ and $t \circ e'^* = e'$ (cf. [10]).



2.1. Proposition. Let $\mathcal{B}$ have pushouts and $V: \mathcal{A} \rightarrow \mathcal{B}$ be a left adjoint to the functor $G: \mathcal{B} \rightarrow \mathcal{A}$ with counit $\delta: V \circ G \rightarrow 1_{\mathcal{B}}$. Then each $\xi: Gb' \rightarrow Gb$ has a G -lifter.

PROOF. Take the following pushout in $\mathcal{B}$. ||

(2./2)

$$\begin{array}{ccc}
 VGb' & \xrightarrow{V\xi} & VGb \xrightarrow{\delta_b} b \\
 \delta_{b'} \downarrow & & \downarrow e^* \\
 b' & \xrightarrow{e'^*} & z^*
 \end{array}$$

2.2. Remark. Clearly $\dashv G$ in the above proposition can be replaced by the following weaker condition: $\overline{\dashv} G$. The latter relative adjointness (for relative adjoints see [16]) is equivalent to the existence of initial objects in the commacategories $(Gb \downarrow G)$ for $b \in |\mathcal{B}|$. ||

2.3. Corollary. Suppose that $\mathcal{A}(F)$ has pushouts and free algebras (i.e. there is a left adjoint to $U: \mathcal{A}(F) \rightarrow \mathcal{A}$), then there exists a U -lifter for all $\xi: U\langle a', h' \rangle \rightarrow U\langle a, h \rangle$. ||

2.4. Theorem. Let $\mathcal{A}$ be a cocomplete $\mathcal{E}$ -cowell powered category with an image factorization system $(\mathcal{E}, \mathcal{M})$. If the functor $F: \mathcal{A} \rightarrow \mathcal{A}$ preserves $\mathcal{E}$ then each morphism $\xi: U\langle a', h' \rangle \rightarrow U\langle a, h \rangle$ with $\xi \in \mathcal{E}$ admits a U -lifter.

PROOF. Let (2./3) represent the $\mathcal{A}$ -colimit of the noncommutative triangle (NC) with the colimit morphisms ε and π .

(2./3)

$$\begin{array}{ccc}
 a & \xrightarrow{\varepsilon} & x \\
 \uparrow \xi & \nearrow h & \uparrow \pi \\
 a' & & \\
 \uparrow h' & & \\
 Fa' & \xrightarrow{(NC)} & Fa
 \end{array}$$

At first we claim that $\varepsilon \in \mathcal{E}$. If $\varepsilon = m \circ e$ is a factorization of ε with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ then there is a unique morphism $p: Fa \rightarrow y$ making (2./4) commute.

(2./4)

$$\begin{array}{ccccc}
 & & y & & \\
 & e \nearrow & & \nwarrow m \in \mathcal{M} & \\
 a & & & & x \\
 \uparrow \xi & & & & \uparrow \pi \\
 a' & & & & \\
 \uparrow h' & & & & \\
 Fa' & \xrightarrow{F\xi \in \mathcal{E}} & Fa & & \\
 & \nwarrow p & \nearrow & &
 \end{array}$$

Since the morphisms e and p create a cocone for (NC) there exists a unique $k: x \rightarrow y$ with $k \circ \varepsilon = e$ and $k \circ \pi = p$. $k \circ m \circ e = k \circ \varepsilon = 1_y \circ e$ implies $k \circ m = 1_y$ and the equations $m \circ k \circ \varepsilon = m \circ e = \varepsilon$, $m \circ k \circ \pi = m \circ p = \pi$ imply $m \circ k = 1_x$. Thus m is proved to be an isomorphism, consequently $\varepsilon \in \mathcal{E}$ by $m \in \mathcal{E}$.

It follows from $F\varepsilon \in \mathcal{E}$ that the free completion (2./5) of the partial F -algebra $x \xleftarrow{\pi} Fa \xrightarrow{F\varepsilon} Fx$ exists (the pushout construction will stop).

(2./5)

$$\begin{array}{ccc}
 \pi \dashv Fa & \xrightarrow{F\varepsilon} & Fx \\
 f^* \downarrow & & \downarrow Ff^* \\
 \omega^* & \xleftarrow{u^*} & F\omega^*
 \end{array}$$

It can easily be seen that $\langle f^* \circ \varepsilon \circ \xi, \langle \omega^*, u^* \rangle, f^* \circ \varepsilon \rangle$ will be the U -lifter of ξ . ||

2.5. Remark. For the basic properties of image factorization systems in categories see [3, 5—6, 11, 13, 15]. The result on free completions of partial F -algebras used in the proof of (2.4) can be found e.g. in [5, 11]. ||

Let $\mathcal{N}$ be the following category. Objects: $|\mathcal{N}| = \{0, 1, 2, \dots, n, \dots\}$. Morphisms: the units, and for each $n \geq 1$ there are n distinct morphisms $d_i^n: n \rightarrow 0$ ($i=0, 1, \dots, n-1$). Composition: obvious.

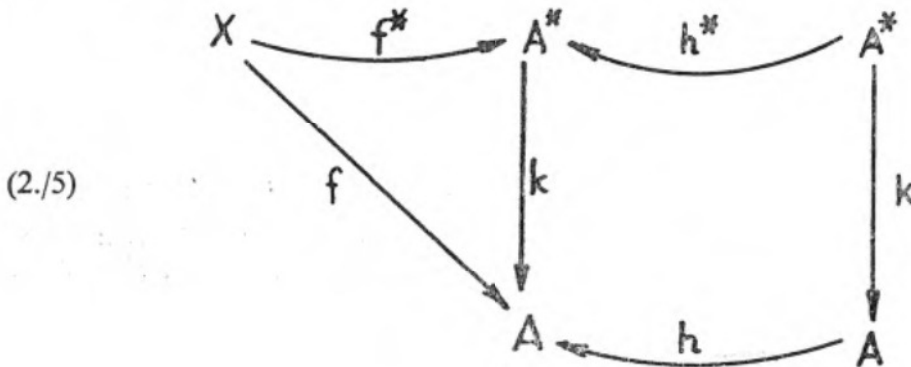
2.6. Theorem. Let $\mathcal{A}$ be $\mathcal{N}$ -cocomplete (each functor $\mathcal{N} \rightarrow \mathcal{A}$ has a colimit in $\mathcal{A}$) and let F preserve these colimits, then each $\xi: U\langle a', h' \rangle \rightarrow U\langle a, h \rangle$ has a U -lifter.

PROOF. Define the functor $N: \mathcal{N} \rightarrow \mathcal{A}$ by $n \rightarrow 0 \mapsto F^{n-1}a' \rightarrow a$ with $Nd_i^n = h \circ Fh \circ \dots \circ F^{i-1}h \circ F^i\xi \circ F^i h' \circ \dots \circ F^{n-2}h'$ ($F^0 = 1_{\mathcal{A}}$ and F^{-1} is irregular). Let $q^*: N \rightarrow \omega^*$ be a colimit cocone then $Fq^*: F \circ N \rightarrow F\omega^*$ is a colimit cocone too. Define $Q: F \circ N \rightarrow \omega^*$ as follows: $Q_0 = q_0^* \circ h$, and for $n \geq 1$ let $Q_n = Q_0 \circ FNd_0^n$. Clearly, Q is a cocone since $h \circ FNd_i^n = Nd_{i+1}^{n+1}$ ($0 \leq i \leq n-1$). Hence there is a unique $u^*: F\omega^* \rightarrow \omega^*$ with $u^* \circ Fq^* = Q$. Now $q_0^*: \langle a, h \rangle \rightarrow \langle \omega^*, u^* \rangle$ and $q_0^* \circ \xi: \langle a', h' \rangle \rightarrow \langle \omega^*, u^* \rangle$ are $\mathcal{A}(F)$ -morphisms because of $q_0^* \circ h = Q_0 = u^* \circ Fq_0^*$ and $(q_0^* \circ \xi) \circ h' = = q_0^* \circ Nd_0^2 = q_0^* \circ Nd_1^2 = q_0^* \circ h \circ F\xi = u^* \circ F(q_0^* \circ \xi)$. We prove that $\langle q_0^* \circ \xi, \langle \omega^*, u^* \rangle, q_0^* \rangle$ is the required U -lifter. Let $\langle e', \langle \omega, u \rangle, e \rangle$ be a triple with $\langle \omega, u \rangle \in |\mathcal{A}(F)|$, $e': \langle a', h' \rangle \rightarrow \langle \omega, u \rangle$, $e: \langle a, h \rangle \rightarrow \langle \omega, u \rangle$ and $(Ue) \circ \xi = Ue'$ then define $q: N \rightarrow \omega$ as follows: $q_0 = e$, and for $n \geq 1$ let $q_n = q_0 \circ Nd_0^n$. q is a cocone: for $0 \leq i \leq n-1$ we have $q_0 \circ Nd_i^n = e \circ h \circ Fh \circ \dots \circ F^{i-1}h \circ F^i\xi \circ F^i h' \circ \dots \circ F^{n-2}h' = u \circ Fu \circ \dots \circ F^{i-1}u \circ F^i e' \circ F^i h' \circ \dots \circ F^{n-2}h' = e' \circ h' \circ Fh' \circ \dots \circ F^{i-1}h' \circ F^i h' \circ \dots \circ F^{n-2}h' = e \circ \xi \circ h' \circ Fh' \circ \dots \circ F^{n-2}h' = q_0 \circ Nd_0^n = q_n$ since $e' = e \circ \xi$ and $e \circ h = u \circ Fe$, $e' \circ h' = u \circ Fe'$. Hence there is a unique $t: \omega^* \rightarrow \omega$ with $t \circ q^* = q$. Then $t \circ q_0^* = q_0 = e$, $t \circ (q_0^* \circ \xi) = e \circ \xi = e'$ and $t: \langle \omega^*, u^* \rangle \rightarrow \langle \omega, u \rangle$ is in $\mathcal{A}(F)$ by $t \circ Q = u \circ Fq$. Moreover t is uniquely determined already by $t \circ q_0^* = e$. $\parallel$

At a first glance theorem (2.6) seems to be weaker than the corollary (2.3). However this is not the case. In the following example a very simple situation will be given for which (2.6) is applicable but (2.3) is not.

2.7. Example. Let Set_{fin} denote the full subcategory of Set consisting of all finite sets and consider $U: \text{Set}_{\text{fin}}(1) \rightarrow \text{Set}_{\text{fin}}$ where $1: \text{Set}_{\text{fin}} \rightarrow \text{Set}_{\text{fin}}$ is identical. For a functor $N: \mathcal{N} \rightarrow \text{Set}_{\text{fin}}$ let $A_n = Nn$ ($n \geq 0$) then the colimit of N is of the form $A_n \rightarrow A_0 \xrightarrow{\text{canonical}} A_0/\Theta$, where Θ can be obtained as an intersection of appropriate equivalences.

Suppose that a nonvoid $X \in \text{Set}_{\text{fin}}$ generates a free algebra $X \xrightarrow{f^*} A^* \xleftarrow{h^*} A^*$ in $\text{Set}_{\text{fin}}(1)$. Let $A = \{0, 1, \dots, n\}$ and $h(i) = i+1$ ($0 \leq i < n$), $h(n) = 0$ with $n = |A^*|$. Let $f: X \rightarrow A$ be a constant (e.g. $f=0$) then there exists a unique $k: A^* \rightarrow A$ making the following diagram commute.



This is a contradiction since for an $a \in A^*$ we have $n+1$ different elements $k(a)$, $hk(a)$, ..., $h^n k(a)$, consequently $a, h^*(a)$, ..., $(h^*)^n(a)$ are different elements in A^* . $\parallel$

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