

Functional equations of sum form

By L. LOSONCZI (Lagos and Debrecen)

1. Introduction

Let Γ_n denote the set of all complete n -ary probability distributions, that is

$$\Gamma_n = \{p = (p_1, \dots, p_n) \mid p_i \geq 0 \quad i = 1, \dots, n; \sum_{i=1}^n p_i = 1\}$$

and $\Gamma_n^0 = \{p = (p_1, \dots, p_n) \mid p_i > 0 \quad i = 1, \dots, n; \sum_{i=1}^n p_i = 1\}$. The characterization of entropies of degree α leads to the functional equation

$$(1) \quad \sum_{i=1}^k \sum_{j=1}^l f(p_i q_j) = \sum_{i=1}^k f(p_i) + \sum_{j=1}^l f(q_j) + \lambda \sum_{i=1}^k f(p_i) \sum_{j=1}^l f(q_j)$$

where $p \in \Gamma_k$, $q \in \Gamma_l$, $\lambda = 2^{1-\alpha} - 1$ is a constant and $f: [0, 1] \rightarrow \mathbb{R}$ is the unknown function (see e.g. ACZÉL—DARÓCZY [3], LOSONCZI [15]). Allowing different functions in (1) we obtain

$$(2) \quad \sum_{i=1}^k \sum_{j=1}^l [f_{ij}(p_i q_j) - q_j g_i(p_i) - p_i h_j(q_j) - \lambda g_i(p_i) h_j(q_j)] = 0$$

for $p \in \Gamma_k$, $q \in \Gamma_l$. In some cases p, q lie in Γ_k^0, Γ_l^0 respectively. Equations like (2) also arise in various characterizations of measures of information depending on two probability distributions.

The aim of this paper is to investigate a wide class of functional equations and obtain general methods for their solution.

We study the equation

$$(3) \quad \sum_{i=1}^k \sum_{j=1}^l F_{ij}(p_i, q) = 0$$

where either $F_{ij}: [0, 1]^2 \rightarrow \mathbb{R}$, $p \in \Gamma_k$, $q \in \Gamma_l$ or $F_{ij}: (0, 1)^2 \rightarrow \mathbb{R}$, $p \in \Gamma_k^0$, $q \in \Gamma_l^0$ and $k, l \geq 3$ are fixed integers. An equation of form (3) is called a *functional equation of sum form*.

In Section 2 we prove some lemmas which serve as our main tool in finding the general solution of (3) in Section 3.

In Section 4 we specialize our results for the equation

$$(4) \quad \sum_{i=1}^k \sum_{j=1}^l [f_{ij}(p_i q_j) - \sum_{t=1}^N g_{it}(p_i) h_{jt}(q_j)] = 0 \quad (p \in \Gamma_k, q \in \Gamma_l)$$

(or $p \in \Gamma_k^0, q \in \Gamma_l^0$). This is the most important (but still quite general) special case of (3). Equation (3) has been solved if all functions F_{ij} are equal and $p \in \Gamma_k, q \in \Gamma_l$ (see LOSONCZI—MAKSA [18]). *Apart from this and a result of Kannappan [13] concerning the measurable solutions of (3) if $p \in \Gamma_k, q \in \Gamma_l$ all sum form equations which have been investigated so far are of form (4).* In some cases however some of the functions g_{it}, h_{jt} are known (KANNAPPAN [11], [12], LOSONCZI—MAKSA [17] (while in other cases the condition $k, l \geq 3$ is not satisfied (DARÓCZY [7], DARÓCZY—JÁRAI [8], LOSONCZI [15], MAKSA [19]).

We show that under measurability conditions (4) can be reduced to the equations

$$(5) \quad \tilde{f}_{ij}(xy) - \sum_{t=1}^N \tilde{g}_{it}(x) \tilde{h}_{jt}(y) = 0 \quad (x, y \in [0, 1])$$

($i=1, \dots, k; j=1, \dots, l$) where $\tilde{f}_{ij}, \tilde{g}_{it}, \tilde{h}_{jt}$ are new unknown functions obtained from f_{ij}, g_{it}, h_{jt} by adding suitable linear functions. Concerning (5) see ACZÉL [1], ACZÉL—CHUNG [2], JÁRAI [10], LOSONCZI [16], MCKIERNAN [20], SZÉKELYHIDI [24], VINCZE [25].

We remark that if the domain is open (i.e. $p \in \Gamma_k^0, q \in \Gamma_l^0$) or the conditions $k, l \geq 3$ are not satisfied only the measurable solutions of some special cases of (4) have been found (KANNAPPAN—NG [14], SAHOO [22], [23], DARÓCZY—JÁRAI [8]). Finding the general solution of equations of type (4) requires individual treatment for each equation (see LOSONCZI—MAKSA [17], [18]).

In Section 5 we show how the general methods of sections 3, 4 work by solving some equations of type (4). Related equations have been investigated by several authors, among others by ACZÉL—DARÓCZY [3], BEHARA—NATH [5], CHAUNDY—MCLEOD [6], KANNAPPAN [12], [13], LOSONCZI [15], MAKSA [19], MITTAL [21].

Through the paper we shall apply, if convenient, the following notations.

I and Δ_n will denote either $[0, 1]$ and Γ_n or $(0, 1)$ and Γ_n^0 .

If f is a real valued function defined on a set G containing I^k and $r, 2r \in I$ then the difference operator D_i^r ($i=1, \dots, k$) is given by

$$\begin{aligned} (D_i^r f)(x_1, \dots, x_i, \dots, x_k) &= \\ &= f(x_1, \dots, x_i, \dots, x_k) + f(x_1, \dots, 2r, \dots, x_k) - 2f(x_1, \dots, r, \dots, x_k) \quad (x_1, \dots, x_k) \in G. \end{aligned}$$

If f is a function of a single variable then we use D^r instead of D_1^r .

It is easy to check that D_i^r is a linear operator having the following properties,

$$D_i^r D_j^s = D_j^s D_i^r$$

$$(D_i^r)^2 = D_i^r$$

and

$$D_i^r f = f$$

if f is additive in the i th variable.

2. Basic lemmas

Our main tool in solving sum form equations in the next lemma which has been proved in a less general form by LOSONCZI—MAKSA [18].

Lemma 1. *Let $k \geq 3$ be a fixed integer, c a constant. The functions $\Phi_i: I \rightarrow R$ ($i=1, \dots, k$, R is the set of reals) satisfy the functional equation*

$$(6) \quad \sum_{i=1}^k \Phi_i(p_i) = c \quad (p \in \Delta_k)$$

if and only if there exists an additive function $a: R \rightarrow R$ and constants b_i ($i=1, \dots, k$) such that

$$(7) \quad \Phi_i(x) = a(x) + b_i \quad (x \in I),$$

$$(8) \quad a(1) + \sum_{i=1}^k b_i = c$$

hold.

PROOF. *Case 1:* $I=(0, 1)$, $\Delta_k = \Gamma_k^0$. Let first $k > 3$. Take an ε from $(0, 1)$ and choose two different indices n, r from the set $\{2, \dots, k\}$. Let $0 < x, y, x+y < \varepsilon$. Substituting into (6)

$$(9) \quad p_1 = x, p_n = y, p_r = \varepsilon - (x+y), p_s = \frac{1-\varepsilon}{k-3} \quad (s \neq 1, n, r)$$

we get

$$(10) \quad \Phi_1(x) + \Phi_n(y) + \Phi_r(\varepsilon - (x+y)) = c_\varepsilon$$

where c_ε is a constant. Since the right hand side of (10) is symmetric in x, y we obtain

$$\Phi_n(x) - \Phi_1(x) = \Phi_n(y) - \Phi_1(y) \quad (0 < x, y, x+y < \varepsilon).$$

Substituting $y_1, y_2 \in (0, \varepsilon)$ here and observing that the left hand side is the same for $x \in (0, \varepsilon - \max\{y_1, y_2\})$ we conclude that

$$\Phi_n(y_1) - \Phi_1(y_1) = \Phi_n(y_2) - \Phi_1(y_2).$$

Since $\varepsilon \in (0, 1)$ is arbitrary

$$(11) \quad \Phi_n(y) - \Phi_1(y) = d_n \quad (y \in (0, 1); n = 2, \dots, k)$$

where d_n 's are constants. Using (11) we obtain from (10)

$$(12) \quad \Phi_1(x) + \Phi_1(y) + \Phi_1(\varepsilon - (x+y)) = e_\varepsilon \quad (0 < x, y, x+y < \varepsilon)$$

where e_ε is a constant. Substituting $x = \frac{\varepsilon}{2}$ then $y = \frac{\varepsilon}{2}$ and deducting from (12) the equations so obtained we get

$$\Phi_1(\varepsilon - (x+y)) = \Phi_1\left(\frac{\varepsilon}{2} - x\right) + \Phi_1\left(\frac{\varepsilon}{2} - y\right) + e_\varepsilon \quad \left(0 < x, y < \frac{\varepsilon}{2}\right).$$

This means that the function

$$\Psi(x) = \Phi_1(x) - e_\varepsilon$$

is additive on the square $S_{\varepsilon/2} = \left\{ (x, y) \mid 0 < x, y < \frac{\varepsilon}{2} \right\}$, that is

$$\Psi(u+v) = \Psi(u) + \Psi(v) \quad \text{if } (u, v) \in S_{\varepsilon/2}.$$

Applying the quasi-extension theorem of Daróczy—Losonczi ([9], Theorem 4) we conclude that there exists a function $a_\varepsilon: R \rightarrow R$ additive on $R \times R$ such that

$$\Psi(u) - 2\Psi(\alpha) = a_\varepsilon(u) - 2a_\varepsilon(\alpha) \quad u \in (0, \varepsilon)$$

$$\Psi(u) - \Psi(\alpha) = a_\varepsilon(u) - a_\varepsilon(\alpha) \quad u \in \left(0, \frac{\varepsilon}{2}\right)$$

where (α, α) is a fixed point of $S_{\varepsilon/2}$. Therefore $\Psi(\alpha) = a_\varepsilon(\alpha)$ and

$$(13) \quad \Phi_1(u) = \Psi(u) + e_\varepsilon = a_\varepsilon(u) + e_\varepsilon \quad \text{if } u \in (0, \varepsilon).$$

Hence, for $0 < \varepsilon_1, \varepsilon_2 < 1$ we have $a_{\varepsilon_1}(u) + e_{\varepsilon_1} = a_{\varepsilon_2}(u) + e_{\varepsilon_2}$ if $u \in (0, \min\{\varepsilon_1, \varepsilon_2\})$ which implies $a_{\varepsilon_1} = a_{\varepsilon_2} = a$, $e_{\varepsilon_1} = e_{\varepsilon_2} = e$ and

$$(14) \quad \Phi_1(u) = a(u) + e$$

for $u \in (0, \varepsilon)$ thus for $u \in (0, 1)$ as well. By (11), (14)

$$\Phi_n(x) = a(x) + b_n \quad (x \in (0, 1), n = 1, \dots, k)$$

which proves (7).

If $k=3$ then instead of (9) we use the substitution

$$p_1 = x, \quad p_n = y, \quad p_r = 1 - (x + y)$$

where $0 < x, y, x+y < 1$. Repeating the calculations above with $\varepsilon=1$ (and with some omissions) we get (7).

It is easy to see that (7) is a solution of (6) if and only if (8) holds.

Case 2: $I=[0, 1]$, $\Delta_k = \Gamma_k$. By Case 1 we have (7) for $x \in (0, 1)$ and (8). Thus we only have to show that $\Phi_i(0) = b_i$, $\Phi_i(1) = a(1) + b_i$ ($i=1, \dots, k$) i.e. (8) holds for $x=0$ and $x=1$. Substituting e.g. $p_1=0$, $(p_2, \dots, p_k) \in \Gamma_{k-1}^0$ into (6) and using (7) for $x \in (0, 1)$ and (8) we get

$$c = \Phi_1(0) + \sum_{i=2}^k [a(p_i) + b_i] = \Phi_1(0) + a(1) + \sum_{i=2}^k b_i$$

that is $\Phi_1(0) = b_1$. Similarly $\Phi_i(0) = b_i$ for $i=2, \dots, k$. With $p_1=1$, $p_2=\dots=p_k=0$ we obtain from (6)

$$\Phi_1(1) + \sum_{i=2}^k b_i = c$$

hence, by (8), $\Phi_1(1) = a(1) + b_1$ and in the same way $\Phi_i(1) = a(1) + b_i$ ($i=2, \dots, k$).

Corollary 1. Equation (6) holds if and only if

$$(15) \quad \Phi_i(x) + \Phi_i(2r) - 2\Phi_i(r) = a(x) \quad (x \in I, i = 1, \dots, k)$$

or if $(D^r \Phi_i)(x) = a(x)$ and

$$(16) \quad a(1) = c + \sum_{i=1}^k [\Phi_i(2r) - 2\Phi_i(r)]$$

holds where r is a fixed value in I such that $2r \in I$.

PROOF. From (7) with $x=2r$ and $x=r$ using $a(2r)=2a(r)$ we easily obtain $b_i=2\Phi_i(r)-\Phi_i(2r)$ hence (7), (8) give (15), (16).

Corollary 2. Suppose that one of the functions $\Phi_1, \dots, \Phi_k$ is measurable. Then (6) holds if and only if

$$(17) \quad \Phi_i(x) - \Phi_i(r) - \frac{c - \sum_{m=1}^k \Phi_m(r)}{1 - kr} (x - r) = 0 \quad (x \in I, i = 1, \dots, k)$$

where $r \in I$ is a fixed number with $kr - 1 \neq 0$.

PROOF. If e.g. Φ_1 is measurable then by (7) so is the additive function a hence (see Aczél [1]) $a(x) = a(1)x$. With $x=r$ from (7) $\Phi_i(x) - \Phi_i(r) = a(1)(x-1)$, calculating $a(1)$ here we get exactly (17).

Lemma 2. Let $k, l \geq 3$ be fixed integers and suppose that the functions $F_{ij}: I^2 \rightarrow R$ ($i=1, \dots, k; j=1, \dots, l$) satisfy the equation

$$(3) \quad \sum_{i=1}^k \sum_{j=1}^l F_{ij}(p_i, q_j) = 0 \quad (p \in \Delta_k, q \in \Delta_l).$$

Then there exist functions $a_j: R \times I \rightarrow R$ ($j=1, \dots, l$) and $b_i: I \times R \rightarrow R$ ($i=1, \dots, k$) such that $a_j(\cdot, y)$, $b_i(x, \cdot)$ are additive for every fixed value of $y, x \in I$ respectively and for every fixed values $r, s \in I$ with $2r, 2s \in I$ we have

$$(18) \quad (D_2^s D_1^r F_{ij})(x, y) = (D_2^s D_1^r F_{11})(x, y) + a_j(x, y) + b_i(x, y)$$

$$(x, y \in I, i = 1, \dots, k; j = 1, \dots, l)$$

moreover $a_1 = b_1 \equiv 0$.

PROOF. Equation (3) can be rewritten as

$$(19) \quad \sum_{i=1}^k \Phi_i(p_i, q_1, \dots, q_l) = 0 \quad (p \in \Delta_k, q \in \Delta_l)$$

where

$$\Phi_i(p, q_1, \dots, q_l) = \sum_{j=1}^l F_{ij}(p, q_j).$$

From corollary 1 formula (15) we get

$$D_1^r \Phi_i - D_1^r \Phi_1 = 0 \quad (i = 1, \dots, k)$$

that is

$$\sum_{j=1}^l [(D_1^r F_{ij})(p, q_j) - (D_1^r F_{1j})(p, q_j)] = 0.$$

Applying corollary 1 again there are functions $b_2: I \times R \rightarrow R$ additive in the second variable such that

$$D_2^s(D_1^r F_{ij} - D_1^r F_{1j})(x, y) = b_i(x, y)$$

or

$$(20) \quad (D_2^s D_1^r F_{ij})(x, y) - (D_2^s D_1^r F_{1j})(x, y) = b_i(x, y)$$

with $b_1 \equiv 0$, $i = 1, \dots, k$.

Interchanging the roles of the indices i, j in the above reasoning we get.

$$(21) \quad (D_2^s D_1^r F_{ij})(x, y) - (D_2^s D_1^r F_{in})(x, y) = a_j(x, y) \quad (j = 1, \dots, l)$$

where a_j 's are as described in lemma 2 and $a_1 \equiv 0$. From (20) and (21) our representation (18) follows immediately.

Lemma 3. *If (3) holds then we have*

$$(22) \quad (D_2^s D_1^r F_{ij})(x, y) = \bar{a}_j(x, y) + \bar{b}_i(x, y) + \bar{c}_{ij}(x, y) \\ (x, y \in I, i = 1, \dots, k; j = 1, \dots, l)$$

where $\bar{a}_j: R \times I \rightarrow R$, $\bar{b}_i: I \times R \rightarrow R$ are functions additive in the first, second variable respectively, $\bar{c}_{ij}: R \times R \rightarrow R$ are biadditive functions (i.e. additive in both variables), r, s are fixed elements of I with $2r, 2s \in I$.

PROOF. Writing (3) in the form (19) and applying corollary 1 we get

$$(23) \quad \sum_{j=1}^l (D_1^r F_{ij})(x, q_j) = a(x, q_1, \dots, q_l)$$

where $a: R \times \Delta_l \rightarrow R$ is additive in the first variable. Let now $H = \{h_\gamma | \gamma \in S\}$ be a Hamel-basis of the real numbers over the field of rationals such that $H \subset I$. If $x = \sum_{\alpha} r_{\alpha}(x) h_{\alpha}(x) = \sum_{\alpha} r_{\alpha} h_{\alpha}$ (r_{α} is rational, $h_{\alpha} \in H$ the summation extends to a finite number of indices depending on x) we have

$$a(x, q_1, \dots, q_l) = \sum_{\alpha} r_{\alpha} a(h_{\alpha}, q_1, \dots, q_l)$$

thus by (23)

$$\sum_{j=1}^l [(D_1^r F_{ij})(x, q_j) - \sum_{\alpha} r_{\alpha} (D_1^r F_{ij})(h_{\alpha}, q_j)] = 0 \quad (i = 1, \dots, k).$$

We apply corollary 1 again (keeping x, i fixed) to get

$$(24) \quad (D_2^s D_1^r F_{ij})(x, y) - \sum_{\alpha} r_{\alpha} (D_2^s D_1^r F_{ij})(h_{\alpha}, y) = \bar{b}_j(x, y) \quad (x, y \in I)$$

where $\bar{b}_j: I \times R \rightarrow R$ is additive in the second variable. Define $a_{ij}: R \times I \rightarrow R$ by the equation

$$(25) \quad a_{ij}(x, y) = \sum_{\alpha} r_{\alpha} (D_2^s D_1^r F_{ij})(h_{\alpha}, y)$$

if $x = \sum_{\alpha} r_{\alpha} h_{\alpha} \in R$, $y \in I$ then a_{ij} is additive in the first variable and by (24), (25)

$$(26) \quad (D_2^s D_1^r F_{ij})(x, y) = a_{ij}(x, y) + \bar{b}_j(x, y) \quad (x, y \in I).$$

Interchanging the roles of the variables p_i, q_j in the above argument we get

$$(27) \quad (D_1^* D_2^* F_{ij})(x, y) = \bar{a}_i(x, y) + b_{ij}(x, y) \quad (x, y \in I)$$

where $\bar{a}_i, b_{ij}$ are additive in the first, second variable respectively.

A comparison of (26), (27) shows that

$$(28) \quad a_{ij}(x, y) - \bar{a}_i(x, y) = b_{ij}(x, y) - \bar{b}_j(x, y) \quad (x, y \in I).$$

Defining $c_{ij}(x, y)$ by the right hand side of this equation we see that $c_{ij}: I \times R \rightarrow R$ is additive in the second variable and by (28) it is also additive in the first variable on the triangle $\{(u, v) | u, v, u+v \in I\}$. Applying theorem 4 of [9] we conclude that there exists a biadditive extension $\bar{c}_{ij}: R \times R \rightarrow R$ of c_{ij} . Since $b_{ij}(x, y) = \bar{b}_j(x, y) + \bar{c}_{ij}(x, y)$ we obtain (22) from (27).

3. The solution of equation (3)

First we determine the measurable solutions of (3).

Theorem 1. Suppose that $k, l \geq 3$ are fixed integers, $F_{ij}: I^2 \rightarrow R$ are functions such that $F_{1j}(\cdot, y)$ ($j=1, \dots, l$), $F_{i1}(x, \cdot)$ ($i=1, \dots, k$) are measurable for every fixed value of $y, x \in I$ respectively. Then (3) holds if and only if

$$(29) \quad \begin{aligned} F_{ij}(x, y) = & F_{ij}(x, s) + F_{ij}(r, y) - F_{ij}(r, s) + \\ & + \frac{x-r}{kr-1} \sum_{u=1}^k [F_{uj}(r, y) - F_{uj}(r, s)] + \frac{y-s}{ls-1} \sum_{v=1}^l [F_{iv}(x, s) - F_{iv}(r, s)] + \\ & + \frac{(x-r)(y-s)}{(kr-1)(ls-1)} \sum_{u=1}^k \sum_{v=1}^l F_{uv}(r, s) \quad (x, y \in I, i=1, \dots, k; j=1, \dots, l) \end{aligned}$$

holds where r, s are fixed numbers in I with $(kr-1)(ls-1) \neq 0$.

In other words, if $F_{i1}(x, \cdot), F_{1j}(\cdot, y)$ are measurable then the general solution of (3) is given by (29) where $x \rightarrow F_{ij}(x, s), y \rightarrow F_{ij}(r, y)$ are arbitrary functions (measurable if $i=1, j=1$ respectively) having the same value $F_{ij}(r, s)$ at $x=r, y=s$ respectively.

PROOF. Writing (3) in the form (19) and applying corollary 2 we obtain

$$\sum_{j=1}^l \left[F_{ij}(x, q_j) - F_{ij}(r, q_j) - \frac{x-r}{kr-1} \sum_{u=1}^k F_{uj}(r, q_j) \right] = 0.$$

Using corollary 2 again we get

$$\begin{aligned} F_{ij}(x, y) - F_{ij}(r, y) - \frac{x-r}{kr-1} \sum_{u=1}^k F_{uj}(r, y) - F_{ij}(x, s) + F_{ij}(r, s) + \frac{x-r}{kr-1} \sum_{u=1}^k F_{uj}(r, s) - \\ - \frac{y-s}{ls-1} \sum_{v=1}^l \left[F_{iv}(x, s) - F_{iv}(r, s) - \frac{x-r}{kr-1} \sum_{u=1}^k F_{uv}(r, s) \right] = 0 \end{aligned}$$

which is exactly (29).

The next theorem gives the *general solution* of (3).

Theorem 2. Let $k, l \geq 3$ be fixed integers, $F_{ij}: I^2 \rightarrow R$ ($i=1, \dots, k$; $j=1, \dots, l$) functions satisfying (3). Then

$$(30) \quad (D_2^s D_1^r F_{ij})(x, y) = A_j(x, y) + B_i(x, y) + C(x, y)$$

or, in detailed form,

$$(31) \quad F_{ij}(x, y) = 2F_{ij}(x, s) - F_{ij}(x, 2s) + 2F_{ij}(r, y) - F_{ij}(2r, y) + \\ + 2F_{ij}(r, 2s) + 2F_{ij}(2r, s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + \\ + A_j(x, y) + B_i(x, y) + C(x, y) \\ (x, y \in I, i = 1, \dots, k; j = 1, \dots, l)$$

where r, s are constants with $r, s, 2r, 2s \in I$ $A_j: R \times I \rightarrow R$ ($j=1, \dots, l$) and $B_i: I \times R \rightarrow R$ ($i=1, \dots, k$) are additive functions in the first and second variable respectively, $C: R \times R \rightarrow R$ is a biadditive function such that there are additive functions $a, b: R \rightarrow R$ with the properties

$$(32) \quad a(y) = (D_2^s A_j)(1, y) + \sum_{i=1}^k [2(D_2^s F_{ij})(r, y) - (D_2^s F_{ij})(2r, y)], \quad (y \in I, j = 1, \dots, l)$$

$$(33) \quad b(x) = (D_1^r B_i)(x, 1) + \sum_{j=1}^l [2(D_1^r F_{ij})(x, s) - (D_1^r F_{ij})(x, 2s)], \quad (x \in I, i = 1, \dots, k)$$

$$(34) \quad a(1) + b(1) + C(1, 1) = \sum_{j=1}^l [A_j(1, 2s) - 2A_j(1, s)] + \sum_{i=1}^k [B_i(2r, 1) - 2B_i(r, 1)] + \\ + \sum_{i=1}^k \sum_{j=1}^l [2F_{ij}(r, 2s) + 2F_{ij}(2r, s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s)].$$

Conversely, functions of the form (31) satisfy (3) if (32), (33), (34) hold.

PROOF. Suppose that (3) holds. By Lemma 2 and Lemma 3 (formulae (18), (22)) we get (30) where $A_j = a_j + \bar{a}_1$, $B_i = b_i + \bar{b}_1$, $C = \bar{c}_{11}$ are functions with the additivity properties stated in theorem 2. (31) is just a reformulation of (30).

Now we show that (31) is a solution of (3) if and only if (32), (33), (34) hold. Substituting (31) into (3) and using the additivity properties of A_j, B_i, C we get

$$(35) \quad \sum_{j=1}^l [A_j(1, q_j) + \sum_{i=1}^k (2F_{ij}(r, q_j) - F_{ij}(2r, q_j))] = \\ = -d - C(1, 1) - \sum_{i=1}^k [B_i(p_i, 1) + \sum_{j=1}^l (2F_{ij}(p_i, s) - F_{ij}(p_i, 2s))]$$

where $d = \sum_{i=1}^k \sum_{j=1}^l D_{ij}$ and

$$(36) \quad D_{ij} = 2F_{ij}(r, 2s) + 2F_{ij}(2r, s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s)$$

is the "constant part" of F_{ij} . By corollary 1, applied for the function

$$\Phi_j(y) = A_j(1, y) + \sum_{i=1}^k [2F_{ij}(r, y) - F_{ij}(2r, y)],$$

(35) holds if and only if there exists an additive function $a: R \rightarrow R$ such that (32) and

$$(37) \quad a(1) = -d - C(1, 1) - \sum_{i=1}^k [B_i(p_i, 1) + \sum_{j=1}^l (2F_{ij}(p_i, s) - F_{ij}(p_i, 2s))] + \\ + \sum_{j=1}^l [A_j(1, 2s) - 2A_j(1, s) + \sum_{i=1}^k (2F_{ij}(r, 2s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + 2F_{ij}(2r, s))]$$

hold. (37) is again an equation of type (6). Applying corollary 1 for (37) we see that it holds if and only if there is an additive function $b: R \rightarrow R$ such that (33) and

$$(38) \quad b(1) = -d - C(1, 1) - a(1) + \\ + \sum_{j=1}^l [A_j(1, 2s) - 2A_j(1, s) + \sum_{i=1}^k (2F_{ij}(r, 2s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + 2F_{ij}(2r, s))] + \\ + \sum_{i=1}^k [B_i(2r, 1) - 2B_i(r, 1) + \sum_{j=1}^l (2F_{ij}(2r, s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + 2F_{ij}(r, 2s))]$$

are valid. Using the definition of D_{ij} , d (38) reduces to (34). Thus we have proved that the functions (31) satisfy (3) if and only if (32), (33), (34) hold.

Remark. If $I=[0, 1]$ we may choose $r=s=0$ in (31) and obtain

$$(31)^* \quad F_{ij}(x, y) = F_{ij}(x, 0) + F_{ij}(0, y) - F_{ij}(0, 0) + A_j(x, y) + B_i(x, y) + C(x, y)$$

as the general solution of (3). ((32), (33), (34) have much simpler forms too).

The following theorem shows how equation (3) can be reduced to the equation

$$\sum_{i=1}^k \sum_{j=1}^l F(p_i, q_j) = 0 \quad (p \in A_k, q \in A_l).$$

Theorem 3. Suppose that $k, l \geq 3$ are fixed integers, $F_{ij}: I^2 \rightarrow R$ are functions satisfying (3) and r, s are constants with $r, s, 2r, 2s \in I$. Let

$$(39) \quad \bar{F}_{ij}(x, y) = (D_2^s D_1^r F_{ij})(x, y) + \\ + x [2F_{ij}(r, 2s) - 2F_{ij}(r, s) + \sum_{u=1}^k (2(D_2^s F_{uj})(r, y) - (D_2^s F_{uj})(2r, y))] + \\ + y [2F_{ij}(2s, r) - 2F_{ij}(r, s) + \sum_{v=1}^l (2(D_1^r F_{iv})(x, s) - (D_1^r F_{iv})(x, 2s))] - \\ - xy [\sum_{u=1}^k (2F_{uj}(2r, s) - 2F_{uj}(r, s)) + \sum_{v=1}^l (2F_{iv}(r, 2s) - 2F_{iv}(r, s)) + \\ + \sum_{u=1}^k \sum_{v=1}^l (2F_{uv}(r, 2s) + 2F_{uv}(2r, s) - F_{uv}(2r, 2s) - 4F_{uv}(r, s))]$$

if $x, y \in I$, $i=1, \dots, k$; $j=1, \dots, l$. Then for all possible fixed indices i, j we have

$$(40) \quad \sum_{m=1}^k \sum_{n=1}^l \bar{F}_{ij}(p_m, q_n) = 0 \quad (p \in \Delta_k, q \in \Delta_l).$$

PROOF. By theorem 2

$$(30) \quad (D_2^s D_1^r F_{ij})(x, y) = A_j(x, y) + B_i(x, y) + C(x, y)$$

and (32), (33), (34) hold. Using the additivity properties of A_j, B_i, C we easily obtain from (30) that

$$(41) \quad A_j(1, 2s) - 2A_j(1, s) = (D_2^s D_1^r F_{ij})(1, 2s) - 2(D_2^s D_1^r F_{ij})(r, s) = \\ = 2F_{ij}(r, 2s) - 2F_{ij}(r, s)$$

and

$$(42) \quad B_i(2r, 1) - 2B_i(r, 1) = 2F_{ij}(2s, r) - 2F_{ij}(r, s).$$

By (32) and (41) the coefficient of x in $\bar{F}_{ij}$ is exactly $a(y) - A_j(1, y)$ while that of y is $b(x) - B_i(x, 1)$. Thus $\bar{F}_{ij}$ can be rewritten as

$$\bar{F}_{ij}(x, y) = A_j(x, y) + B_i(x, y) + C(x, y) + x[a(y) - A_j(1, y)] + y[b(x) - B_i(x, 1)] - \\ - xy \left[\sum_{u=1}^k (B_u(2r, 1) - 2B_u(r, 1)) + \sum_{v=1}^l (A_v(1, 2s) - 2A_v(1, s)) + \sum_{u=1}^k \sum_{v=1}^l D_{uv} \right]$$

where D_{uv} is the expression defined by (36). If we use the additivity properties of A_j, B_i, C, a, b we get

$$\sum_{m=1}^k \sum_{n=1}^l \bar{F}_{ij}(p_m, q_n) = \sum_{n=1}^l A_j(1, q_n) + \sum_{m=1}^k B_i(p_m, 1) + C(1, 1) + \\ + a(1) - \sum_{n=1}^l A_j(1, q_n) + b(1) - \sum_{m=1}^k B_i(p_m, 1) - \\ - \sum_{u=1}^k (B_u(2r, 1) - 2B_u(r, 1)) - \sum_{v=1}^l (A_v(1, 2s) - 2A_v(1, s)) - \sum_{u=1}^k \sum_{v=1}^l D_{uv} = 0$$

which completes the proof.

4. The investigation of equation (4)

Equation (4) can be obtained from (3) by choosing

$$(43) \quad F_{ij}(x, y) = f_{ij}(xy) - \sum_{t=1}^N g_{it}(x) h_{jt}(y) \quad (x, y \in I).$$

Theorem 4. Let $k, l \geq 3$ be fixed integers and assume that the functions f_{ij} , f_{i1} , g_{it} , h_{jt} ($i=1, \dots, k$; $j=1, \dots, l$; $t=1, \dots, N$) are measurable. Then

$$(4) \quad \sum_{i=1}^k \sum_{j=1}^l [f_{ij}(p_i q_j) - \sum_{t=1}^N g_{it}(p_i) h_{jt}(q_j)] = 0 \quad (p \in \Delta_k, q \in \Delta_l)$$

holds if and only if

$$(44) \quad \tilde{F}_{ij}(x, y) - \sum_{t=1}^N \tilde{g}_{it}(x) \tilde{h}_{jt}(y) = 0 \quad (x, y \in I; i = 1, \dots, k; j = 1, \dots, l)$$

where

$$(45) \quad \begin{aligned} \tilde{F}_{ij}(x, y) = & f_{ij}(xy) - f_{ij}(xs) - f_{ij}(ry) + f_{ij}(rs) + \\ & + \frac{x-r}{1-kr} \sum_{u=1}^k (f_{uj}(ry) - f_{uj}(rs)) + \frac{y-s}{1-ls} \sum_{v=1}^l (f_{iv}(xs) - f_{iv}(rs)) + \\ & + \frac{(x-r)(y-s)}{(1-kr)(1-lr)} \sum_{u=1}^k \sum_{v=1}^l f_{uv}(rs), \end{aligned}$$

$$(46) \quad \tilde{g}_{it}(x) = g_{it}(x) - g_{it}(r) + \frac{x-r}{1-kr} \sum_{u=1}^k g_{ut}(r),$$

$$(47) \quad \begin{aligned} \tilde{h}_{jt}(x) = & h_{jt}(x) - h_{jt}(s) + \frac{x-s}{1-ls} \sum_{v=1}^l h_{vt}(s) \\ & (i = 1, \dots, k; j = 1, \dots, l; t = 1, \dots, N; x, y \in I) \end{aligned}$$

r, s are being arbitrary elements of I with $kr \neq 1, ls \neq 1$.

PROOF. Applying theorem 1 for the function (43) we immediately obtain (44).

Remarks 1. If $I=[0, 1]$, $\Delta_n = \Gamma_n$ we may choose $r=s=0$. Then (44) reduces to

$$(48) \quad \tilde{f}_{ij}(xy) - \sum_{t=1}^N \tilde{g}_{it}(x) \tilde{h}_{jt}(y) = 0 \quad (x, y \in I; i = 1, \dots, k; j = 1, \dots, l)$$

where

$$(49) \quad \tilde{f}_{ij}(x) = f_{ij}(x) - f_{ij}(0) + x \sum_{u=1}^k \sum_{v=1}^l f_{uv}(0)$$

and $\tilde{g}_{it}, \tilde{h}_{jt}$ are obtained from (46), (47) with $r=0, s=0$.

2. Since (44) holds for every fixed pair (i, j) of indices it is also an equation of type

$$f(xy) - \sum_{t=1}^N g_t(x) h_t(y) = 0 \quad (x, y \in I).$$

For the sake of simplicity we specialize theorem 3 only in the case $I=[0, 1]$.

Theorem 5. If $k, l \geq 3$ are fixed integers and (4) holds for all $p \in \Gamma_k, q \in \Gamma_l$ then for all possible fixed pair of indices (i, j) we have

$$(50) \quad \sum_{m=1}^k \sum_{n=1}^l [\tilde{f}_{ij}(p_m q_n) - \sum_{t=1}^N \tilde{g}_{it}(p_m) \tilde{h}_{jt}(q_n)] = 0 \quad (p \in \Gamma_k, q \in \Gamma_l)$$

where $\tilde{f}_{ij}$ is given by (49), and $\tilde{g}_{it}, \tilde{h}_{jt}$ are obtained from (40), (47) by substituting $r=s=0$.

PROOF. We may choose $r=s=0$ hence (39) goes over into

$$\begin{aligned} \bar{F}_{ij}(x, y) &= F_{ij}(x, y) - F_{ij}(x, 0) - F_{ij}(0, y) + F_{ij}(0, 0) + \\ &+ x \sum_{u=1}^k [F_{uj}(0, y) - F_{uj}(0, 0)] + y \sum_{v=1}^l [F_{iv}(x, 0) - F_{iv}(0, 0)] + xy \sum_{u=1}^k \sum_{v=1}^l F_{uv}(0, 0). \end{aligned}$$

Now F_{ij} has the form (43) thus

$$\bar{F}_{ij}(x, y) = \bar{f}_{ij}(xy) - \sum_{t=1}^N \bar{g}_{it}(x) \bar{h}_{jt}(y).$$

The proof is completed by applying theorem 3.

5. Applications

First we find the *measurable solutions* of (2). Introducing the functions

$$(51) \quad F_{ij}(x) = x + \lambda f_{ij}(x), \quad G_i(x) = x + \lambda g_i(x), \quad H_j(x) = x + \lambda h_j(x)$$

$x \in [0, 1]$; $i = 1, \dots, k$; $j = 1, \dots, l$ equation (2) can be written as

$$(52) \quad \sum_{i=1}^k \sum_{j=1}^l [F_{ij}(p_i q_j) - G_i(p_i) H_j(q_j)] = 0 \quad (p \in \Gamma_k, q \in \Gamma_l)$$

provided that $\lambda \neq 0$. If f_{ij}, g_i, h_j are measurable then by theorem 4 (52) holds if and only if

$$(53) \quad \bar{F}_{ij}(xy) = \bar{G}_i(x) \bar{H}_j(y) \quad (x, y \in [0, 1]; i = 1, \dots, k; j = 1, \dots, l)$$

where by (46), (47) (with $r=s=0$), (49), (51)

$$\begin{aligned} \bar{F}_{ij}(x) &= x + \lambda f_{ij}(x) - \lambda f_{ij}(0) + \lambda x \sum_{u=1}^k \sum_{v=1}^l f_{uv}(0), \\ (54) \quad \bar{G}_i(x) &= x + \lambda g_i(x) - \lambda g_i(0) + \lambda x \sum_{u=1}^k g_u(0), \\ \bar{H}_j(x) &= x + \lambda h_j(x) - \lambda h_j(0) + \lambda x \sum_{v=1}^l h_v(0). \end{aligned}$$

Lemma 4. If $f, g, h: [0, 1] \rightarrow R$ satisfy the functional equation

$$(55) \quad f(xy) = g(x)h(y) \quad (x, y \in [0, 1])$$

and g, h are measurable, $g(0)=h(0)=0$ then

$$(56) \quad f(x) = 0 \quad x \in [0, 1], \quad g(x) = \begin{cases} 0 & x \in E \\ g(x) & x \in F, \end{cases} \quad h(x) = \begin{cases} \eta(x) & x \in E \\ 0 & x \in F \end{cases}$$

or

$$(57) \quad f(x) = \begin{cases} 0 & x \in [0, 1) \\ ab & x = 1, \end{cases} \quad g(x) = \begin{cases} 0 & x \in [0, 1) \\ a & x = 1, \end{cases} \quad h(x) = \begin{cases} 0 & x \in [0, 1) \\ b & x = 1 \end{cases}$$

or

$$(58) \quad f(x) = \begin{cases} abx^\delta & x \in (0, 1] \\ 0 & x = 0, \end{cases} \quad g(x) = \begin{cases} ax^\delta & x \in (0, 1] \\ 0 & x = 0, \end{cases} \quad h(x) = \begin{cases} bx^\delta & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

where E, F are disjoint measurable sets whose union is $[0, 1]$, $0 \in E$, ϑ, η are arbitrary measurable functions on E, F respectively such that $\vartheta(x) \neq 0$ $x \in E$, $\eta(0) = 0$, $\delta, a \neq 0$, $b \neq 0$ are arbitrary constants.

PROOF. If $g(1) = 0$ or $h(1) = 0$ then (55) shows that $f(x) = 0$ on $[0, 1]$. Let $E = \{x \in [0, 1] \mid g(x) = 0\}$ then $0 \in E$, E is measurable. h is arbitrary on E and zero on $F = [0, 1] - E$ since on F $g(x) \neq 0$. Then $h(0) = 0$ and g is arbitrary on F . This gives solution (56).

If $g(1)h(1) \neq 0$ then the substitutions $x=1$ and $y=1$ give $h(y) = f(y)/g(1)$ and $g(x) = f(x)/h(1)$ thus $\bar{f}(x) = f(x)/g(1)h(1)$ satisfies

$$(59) \quad \bar{f}(xy) = \bar{f}(x)\bar{f}(y) \quad (x, y \in [0, 1]).$$

We have $\bar{f}(1) = f(1)/g(1)h(1) = 1$. We claim that either $\bar{f}(x) = 0$ if $x \in (0, 1)$ or $\bar{f}(x) \neq 0$ if $x \in (0, 1)$. Namely if there exists an $x_0 \in (0, 1)$ with $\bar{f}(x_0) = 0$ then by (59) $\bar{f}(x) = 0$ for $x \in (0, x_0]$. For any $y \in (0, 1)$ we can find an $x \in (0, x_0]$ and a positive integer n such that $y^n = x$. Hence $0 = \bar{f}(x) = \bar{f}(y^n) = [\bar{f}(y)]^n$ and $\bar{f}(y) = 0$.

If $\bar{f}(x) = 0$ $x \in [0, 1)$ then we obtain solution (57) with $a = g(1)$, $b = h(1)$.

If $\bar{f}(x) \neq 0$ $x \in (0, 1]$ then $\bar{f}(x) = \bar{f}(\sqrt{x})^2 > 0$ for $x \in (0, 1]$. By (59) the function $F(t) = \ln \bar{f}(e^{-t})$ $t \in [0, \infty)$ is measurable and additive in the first quadrant:

$$F(t+s) = F(t) + F(s) \quad t, s \geq 0.$$

Thus (see ACZÉL [1], ACZÉL—ERDŐS [4]) F is linear and $\bar{f}(x) = x^\delta$ if $x \in (0, 1]$, $\bar{f}(1) = 1$. This gives solution (58) with $a = g(1)$, $b = h(1)$.

Theorem 6. Let $k, l \geq 3$ be fixed integers f_{i1}, f_{1j}, g_i, h_j ($i=1, \dots, k$; $j=1, \dots, l$) be measurable functions and suppose that (2) holds for all $p \in \Gamma_k$, $q \in \Gamma_l$ and $\lambda \neq 0$. Then

$$f_{ij}(x) = -\frac{x}{\lambda} + \alpha_{ij} - x \sum_{u=1}^k \sum_{v=1}^l \alpha_{uv} + \frac{1}{\lambda} \bar{F}_{ij}(x),$$

$$g_i(x) = -\frac{x}{\lambda} + \beta_i - x \sum_{u=1}^k \beta_u + \frac{1}{\lambda} \bar{G}_i(x),$$

$$h_j(x) = -\frac{x}{\lambda} + \gamma_j - x \sum_{v=1}^l \gamma_v + \frac{1}{\lambda} \bar{H}_j(x)$$

$x \in [0, 1]$; $i=1, \dots, k$; $j=1, \dots, l$ where $\alpha_{ij}, \beta_i, \gamma_j$ are arbitrary constants and $\bar{F}_{ij}, \bar{G}_i, \bar{H}_j$ are given in the following way. All possible pairs (i, j) of indices can be divided in three disjoint sets A, B, C such that if $(i, j) \in A$ then

$$\bar{F}_{ij}(x) = 0 \quad x \in [0, 1], \quad \bar{G}_i(x) = \begin{cases} 0 & x \in E_i \\ \vartheta_i(x) & x \in F_i, \end{cases} \quad \bar{H}_j(x) = \begin{cases} \eta_j(x) & x \in E_i \\ 0 & x \in F_i \end{cases}$$

if $(i, j) \in B$ then

$$\bar{F}_{ij}(x) = \begin{cases} 0 & x \in [0, 1) \\ a_i b_j & x = 1, \end{cases} \quad \bar{G}_i(x) = \begin{cases} 0 & x \in [0, 1) \\ a_i & x = 1, \end{cases} \quad \bar{H}_j(x) = \begin{cases} 0 & x \in [0, 1) \\ b_j & x = 1 \end{cases}$$

if $(i, j) \in C$ then

$$\bar{F}_{ij}(x) = \begin{cases} a_i b_j x^{\delta_{ij}} & x \in (0, 1] \\ 0 & x = 0, \end{cases} \quad \bar{G}_i(x) = \begin{cases} a_i x^{\delta_{ij}} & x \in (0, 1] \\ 0 & x = 0, \end{cases} \quad \bar{H}_j(x) = \begin{cases} b_j x^{\delta_{ij}} & x \in (0, 1] \\ 0 & x = 0. \end{cases}$$

Here E_i, F_i are disjoint measurable sets whose union is $[0, 1]$, $0 \in E_i$, η_i, η_j are arbitrary measurable functions on E_i, F_i respectively such that $\eta_i(x) \neq 0$ on E_i , $\eta_j(0) = 0$. Depending on A the some sets E_i may have to be equal, $a_i \neq 0$, $b_j \neq 0$, δ_{ij} are arbitrary constants. Depending on C some constants δ_{ij} may have to be equal.

The proof is immediate if we apply lemma 4 for equation (53).

As second example we solve the equation

$$(60) \quad \sum_{i=1}^k \sum_{j=1}^l f(p_i q_j) = 0 \quad (p \in \Gamma_k^0, q \in \Gamma_l^0)$$

assuming the measurability of f .

Theorem 7. If $f: (0, 1) \rightarrow R$ is measurable, $k, l \geq 3$ and (60) holds then

$$(61) \quad f(x) = a \left(x - \frac{1}{kl} \right) \quad (x \in (0, 1))$$

where a is a constant.

PROOF. By theorem 4 (60) holds if and only if

$$(62) \quad f(xy) = f(xs) \frac{ly-1}{ls-1} + f(ry) \frac{kx-1}{kr-1} - f(rs) \frac{kx-1}{kr-1} \frac{ly-1}{ls-1}$$

for all $x, y \in [0, 1)$, $r, s \in (0, 1)$, $kr \neq 1$, $ls \neq 1$. With $x = \frac{1}{k}$ we obtain from (62)

$$f(y/k) = \frac{f(s/k)}{ls-1} (ly-1) = a_1 y + b_1 \quad (y \in (0, 1))$$

or $f(y) = ay + b$ if $y \in (0, \frac{1}{k})$. If $r, s < \min \left\{ \frac{1}{k}, \frac{1}{l} \right\}$ then $xs, ry \in (0, \frac{1}{k})$ thus (62) gives

$$f(xy) = Axy + Bx + Cy + D \quad (x, y \in (0, 1))$$

where A, B, C, D are constants. Since f is linear on $(0, \frac{1}{k})$ we get $A = a$, $B = C = 0$, $D = b$ and

$$f(x) = ax + b \quad x \in (0, 1).$$

This is a solution of (60) if and only if $b = -\frac{1}{kl} a$.

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