

Lattice theoretical characterizations of quantum probability space I

By I. G. KALMÁR (Debrecen)

1. The notion of quantum probability space was introduced by SUPPES [6], and later it was investigated by many authors, for example by GUDDER [2].

Let Ω be a nonempty set and let S_σ be a collection of its subsets which satisfy

- (i) $\Omega \in S_\sigma$
- (ii) if $A \in S_\sigma$, then $A^c = \Omega \setminus A \in S_\sigma$
- (iii) if $A_i \in S_\sigma$ ($i = 1, 2, \dots$), $A_i \cap A_j = \emptyset$ ($i \neq j$), then $\bigcup_i A_i \in S_\sigma$

Then we call S_σ a σ -class, and (Ω, S_σ, p) is called a *quantum probability space*, if p is a nonnegative set function on S_σ such that $p(\Omega)=1$, and $p(\bigcup_i A_i) = \sum_i p(A_i)$ if $A_i \cap A_j = \emptyset$ ($i \neq j$).

It is easy to see that quantum probability space is more general than classical probability space, but less general than the usual quantum logic. So the first question that arises: which orthomodular σ -poset (partially ordered set) will be isomorphic to a σ -class?

A simple theorem of Gudder gives a characterization of this type of orthomodular σ -poset with the help of two-valued measures on it. See, for example [3], Theorem 3.28. In this paper we shall give lattice theoretical characterizations of those orthomodular lattices that are isomorphic to a class S of a nonempty set Ω with the following properties:

- (i) $\Omega \in S$
- (ii) if $A \in S$, then $A^c = \Omega \setminus A \in S$
- (iii) if $A_1, A_2 \in S$, $A_1 \cap A_2 = \emptyset$, then $A_1 \cup A_2 \in S$

Such an S will be called an *n-class*.*)

First we recall the basic notions that we shall use.

2. Let $\mathcal{L}(\vee, \wedge, \perp, 0, 1)$ be a complemented lattice with least and greatest elements 0 and 1, respectively. If the complementation $\perp$ satisfies also (i) $(a^\perp)^\perp = a \forall a \in \mathcal{L}$, and (ii) $b^\perp \leq a^\perp$ if $a \leq b$, $a, b \in \mathcal{L}$, then we call $\perp$ *orthocomplementa-*

*) The isomorphism to a σ -class will be examined in Part II.

tion and $\mathcal{L}$ orthocomplemented lattice. We remark that the De Morgan laws are valid in every orthocomplemented lattice.

A complemented lattice is *weakly modular* if

$$a \vee b = a \vee (a^\perp \wedge (a \vee b)) \quad \text{for all } a, b \in \mathcal{L}.$$

An *orthomodular lattice* is an orthocomplemented and weakly modular lattice. The relation of *orthogonality* ($\perp$) for elements a, b of an orthocomplemented lattice is defined by $a \perp b$ if $a \leq b^\perp$. For elements a, b of an orthocomplemented lattice $\mathcal{L}$ we say that they are *compatible*, in symbols $a \leftrightarrow b$, if there exists a Boolean subalgebra in $\mathcal{L}$ containing a and b .

In the following lemma we collected the most important properties of compatibility.

Lemma 1. *In every orthomodular lattice $\mathcal{L}$ the following implications are true:*

(a) *if one of the three elements a, b, c of $\mathcal{L}$ is compatible with each of the two others, then*

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

(b) *if $a \leq b$ or $a \perp b$, then $a \leftrightarrow b$*

(c) *if $a \leftrightarrow b$, then $a \leftrightarrow b^\perp$*

(d) *$a \leftrightarrow b \Leftrightarrow (a \vee b^\perp) \wedge b = a \wedge b$*

(e) *$\mathcal{L}$ is a Boolean algebra $\Leftrightarrow a \leftrightarrow b$ for all $a, b \in \mathcal{L}$*

(f) *$a \leftrightarrow b_i \Rightarrow a \leftrightarrow \bigvee_i b_i$, $a \leftrightarrow \bigwedge_i b_i$ if $\bigvee_i b_i$ and $\bigwedge_i b_i$ exist*

Let $\mathcal{L}$ be a complemented lattice. Let us introduce the *star-product* by

$$a * b = (a \vee b^\perp) \wedge b; \quad a, b \in \mathcal{L}.$$

By an elementary computation one can prove the following

Lemma 2. (See also Proposition 3 in [4]). *If $\mathcal{L}$ is an orthomodular lattice and $a, b, c \in \mathcal{L}$, then*

(a) $a * a = a$

(b) $c * a = 0$ if $c \perp a$

(c) $(c * a) * b = 0$ if $a \perp b$

(d) $(c * a) * a = c * a$

(e) $c * (c * a) = c * a$

(f) $c * a = c \wedge a \Leftrightarrow c \leftrightarrow a$

(g) $(c * a) * b = (c * b) * a = c * a$ if $a \leq b$

(h) $(\bigvee_i c_i) * a = \bigvee_i (c_i * a)$ if $c_i \in \mathcal{L}$ and $\bigvee_i c_i, \bigvee_i (c_i * a)$ exist

(i) $(a * b) * c = a * (b * c)$ if $c \leftrightarrow b$

Let $\mathcal{L}$ be an orthocomplemented lattice. A mapping m of $\mathcal{L}$ into the real line $\mathbb{R}$ is a *probability measure* if (i) $0 \leq m(a) \leq 1 \quad \forall a \in \mathcal{L}$, (ii) $m(1) = 1$, (iii) $m(a \vee b) = m(a) + m(b)$, if $a, b \in \mathcal{L}$, and $a \perp b$.

A probability measure m is a *probability σ -measure* if $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ for all pairwise orthogonal $a_i \in \mathcal{L}$, provided that $\bigvee_{i=1}^{\infty} a_i$ exists. We call a probability measure (probability σ -measure) a *two-valued measure* (*two-valued σ -measure*), if it has only two values 0 and 1.

3. We need some new concepts before going on.

Definition. Let $\mathcal{L}$ be a complemented lattice. A nonempty subset $\mathcal{N}$ of $\mathcal{L}$ will be called a *proper $*$ -filter* if

- (i) $0 \in \mathcal{N}$
- (ii) if $a \in \mathcal{N}$ and $a \leq b$, then $b \in \mathcal{N}$
- (iii) if $a, b \in \mathcal{N}$, then $a * b \in \mathcal{N}$

If moreover $\mathcal{N}$ satisfies also

- (iv) $a \in \mathcal{N}$ or $a^\perp \in \mathcal{N}$ for every $a \in \mathcal{L}$,

then $\mathcal{N}$ is called a *$*$ -ultrafilter*.

Of course, if $\mathcal{N}$ is a $*$ -ultrafilter, then it is not a proper subset of a proper $*$ -filter of $\mathcal{L}$, so $\mathcal{N}$ is maximal in this sense.

Lemma 3. *If $\mathcal{L}$ is an orthomodular lattice, then a nonempty subset $\mathcal{N}$ of $\mathcal{L}$ is a proper $*$ -filter if and only if*

- (i) $0 \in \mathcal{N}$,
- (ii) if $a \in \mathcal{N}$ and $a \leq b$, then $b \in \mathcal{N}$,
- (iii)' if $a, b \in \mathcal{N}$ and $a \leftrightarrow b$, then $a \wedge b \in \mathcal{N}$.

PROOF. If $\mathcal{N}$ is a proper $*$ -filter, then (i) and (ii) hold by definition. If $a, b \in \mathcal{N}$ and $a \leftrightarrow b$, then $a * b = a \wedge b \in \mathcal{N}$. Conversely, if $\mathcal{N}$ satisfies (i), (ii) and (iii)', then $a, b \in \mathcal{N}$ implies $a \vee b^\perp \in \mathcal{N}$. However, $a \vee b^\perp \leftrightarrow b$, so $a * b = (a \vee b^\perp) \wedge b \in \mathcal{N}$, i.e. $\mathcal{N}$ is a proper $*$ -filter, which was to be proved.

A *block* of a complemented lattice $\mathcal{L}$ is a maximal Boolean subalgebra of.

Definition. Let $\hat{\mathbf{B}}_j$, $j \in \mathcal{J}$ be the class of the blocks of a complemented lattice $\mathcal{L}$. Let $\mathcal{J}' \subseteq \mathcal{J}$ and let us suppose that for every $j \in \mathcal{J}'$ there is a maximal filter $\mathcal{N}_j$ in $\hat{\mathbf{B}}_j$ such that

- (i) $\bigcup_{j \in \mathcal{J}'} \mathcal{N}_j$ does not contain orthogonal elements
- (ii) $\bigcup_{j \in \mathcal{J}'} \mathcal{N}_j$ is maximal, that is for every $i \in \mathcal{J} \setminus \mathcal{J}'$

and for every maximal filter $\mathcal{N}_i$ of $\hat{\mathbf{B}}_i$, $(\bigcup_{j \in \mathcal{J}'} \mathcal{N}_j) \cup \mathcal{N}_i$ has orthogonal elements. In this case $\mathcal{R} = \bigcup_{j \in \mathcal{J}'} \mathcal{N}_j$ will be called a *realization*.

If in the preceding definition $\mathcal{J}' = \mathcal{J}$, then we say that $\mathcal{R} = \bigcup_{j \in \mathcal{J}} \mathcal{N}_j$ is a *complete realization*.

Between the notions of $*$ -ultrafilter and complete realization there is a close connection.

Lemma 4. If $\mathcal{L}$ is an orthomodular lattice and $\mathcal{R} \subset \mathcal{L}$, then the following two statements are equivalent:

- (i) $\mathcal{R}$ is a complete realization
- (ii) $\mathcal{R}$ is a $*$ -ultrafilter

PROOF. Let $\mathcal{R} = \bigcup_{j \in \mathcal{J}} \mathcal{N}_j$ be a complete realization. Then a) $0 \notin \mathcal{R}$, b) If $a \in \mathcal{R}$ and $a \leq b$, then there exists a block $\hat{\mathbf{B}}_j$, $j \in \mathcal{J}$ in $\mathcal{L}$ such that $a, b \in \hat{\mathbf{B}}_j$. If $a \in \mathcal{N}_j$, then $a^\perp \in \mathcal{N}_j$, which contradicts $a \in \mathcal{R}$, so $a \in \mathcal{N}_j$. Hence $b \in \mathcal{N}_j$. c) If $a, b \in \mathcal{R}$, then by the preceding property $a \vee b^\perp \in \mathcal{R}$, and from $a \vee b^\perp \leftrightarrow b$ there exists a block $\hat{\mathbf{B}}_k$, $k \in \mathcal{J}$ such that $a \vee b^\perp, b \in \hat{\mathbf{B}}_k$. Then $a \vee b^\perp, b \in \mathcal{N}_k \subset \hat{\mathbf{B}}_k$ and consequently $(a \vee b^\perp) \wedge b = a * b \in \mathcal{N}_k \subset \mathcal{R}$. d) If $c \in \mathcal{L}$, then $c \in \mathcal{R}$ or $c^\perp \in \mathcal{R}$.

a), b), c) and d) mean exactly that $\mathcal{R}$ is a $*$ -ultrafilter.

Conversely, let us suppose that $\mathcal{R}$ is a $*$ -ultrafilter in $\mathcal{L}$. Let $\hat{\mathbf{B}}_j$, $j \in \mathcal{J}$ be the blocks of $\mathcal{L}$ and let $\mathcal{N}_j = \mathcal{R} \cap \hat{\mathbf{B}}_j$, $j \in \mathcal{J}$. Then $\mathcal{N}_j$ is a maximal filter in $\hat{\mathbf{B}}_j$ and $\mathcal{R} = \bigcup_{j \in \mathcal{J}} \mathcal{N}_j$ is a complete realization in $\mathcal{L}$.

Lemma 5. Let $\mathcal{L}$ be an orthocomplemented lattice. Then

- a) if m is a two-valued measure on $\mathcal{L}$, then $\mathcal{R} = \{a \in \mathcal{L} | m(a) = 1\}$ is a $*$ -ultrafilter
- b) if $\mathcal{R}$ is a $*$ -ultrafilter, then $m: \mathcal{L} \rightarrow \{0, 1\}$ defined by

$$m(a) = \begin{cases} 1, & a \in \mathcal{R} \\ 0, & a \in \mathcal{L} \setminus \mathcal{R} \end{cases}$$

is a two-valued measure on $\mathcal{L}$.

PROOF. The proof is somewhat trivial and so it will be not presented here.

Definitions. Let $\mathcal{L}$ be a complemented lattice and denote by G a class of subsets of $\mathcal{L}$. We say that G is *order determining* if $a \in A \Rightarrow b \in A$ for all $A \in G$ implies $a \leq b$. Similarly, a set $\mathfrak{M}$ of probability measures on an orthocomplemented lattice is *order determining* if $m(a) \leq m(b)$ for all $m \in \mathfrak{M}$ implies $a \leq b$.

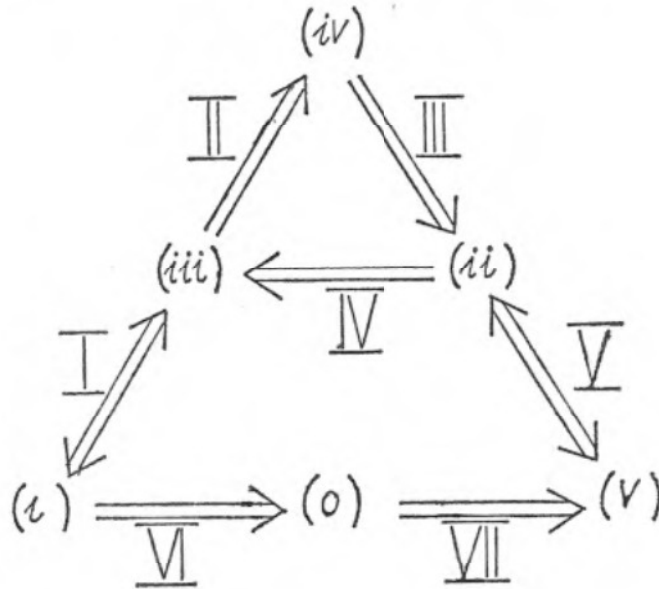
Now we prove our theorem characterizing orthomodular lattices isomorphic to an n -class:

Theorem. Let $\mathcal{L}$ be a complemented lattice. Then the following six statements are equivalent.

- (0) $\mathcal{L}$ is orthomodular and isomorphic to an n -class.
- (i) $\mathcal{L}$ has the following three properties:

- (A) If $a \in \mathcal{L}$, $a \neq 0$, then there exists a complete realization of $\mathcal{L}$ containing a .
 (B) The set of the complete realizations of $\mathcal{L}$ is order determining.
 (C) If $a, b \in \mathcal{L}$ and $a \leq b$, then there exists a Boolean subalgebra of $\mathcal{L}$ which contains a and b .
 (ii) $\mathcal{L}$ has the following two properties:
 (A') If $a \in \mathcal{L}$, $a \neq 0$, then there exists a $*$ -ultrafilter which contains a .
 (B') The set of the $*$ -ultrafilters of $\mathcal{L}$ is order determining.
 (iii) $\mathcal{L}$ is orthomodular and
 (D) for all $a, b \in \mathcal{L}$, $a \perp b$ there exists a complete realization of $\mathcal{L}$ which contains a and b .
 (iv) $\mathcal{L}$ is orthomodular and
 (D') for all $a, b \in \mathcal{L}$, $a \perp b$ there exists a $*$ -ultrafilter of $\mathcal{L}$ which contains a and b .
 (v) The complementation of $\mathcal{L}$ is an orthocomplementation and the set of two-valued probability measures on $\mathcal{L}$ is order determining.

PROOF. The sketch of the proof is the following



I. (i) $\Rightarrow$ (iii) because (C) implies that $\mathcal{L}$ is orthomodular and if $a, b \in \mathcal{L}$, $a \perp b$, then there exists a complete realization of $\mathcal{L}$ containing a . If every such complete realization $\mathcal{R}$ contained also $b^\perp$, then from (B) we should get $a \leq b^\perp$. This contradicts $a \perp b$, so there exists a complete realization of $\mathcal{L}$ containing a and b .

Conversely, (iii) $\Rightarrow$ (i), because the orthomodularity implies (C). Furthermore, if $a \in \mathcal{L}$, $a \neq 0$, then there exists a complete realization containing a and 1, so (A) holds. Finally, if $a \in \mathcal{R} \Rightarrow b \in \mathcal{R}$ for every complete realization $\mathcal{R}$ of $\mathcal{L}$, then there exists no complete realization containing a and b . Therefore $a \perp b^\perp$, that is $a \leq b$. So (B) holds, too.

II. (iii) $\Rightarrow$ (iv) follows immediately from Lemma 4.

III. Let us assume that $\mathcal{L}$ is orthomodular and (D') is satisfied in $\mathcal{L}$. Then (A') holds. If $a, b \in \mathcal{L}$ and $a \in \mathcal{R} \Rightarrow b \in \mathcal{R}$ for all $*$ -ultrafilters $\mathcal{R}$ containing a , then

there exists no $*$ -ultrafilter containing a and $b^\perp$. Hence by (D') $a \perp b^\perp$, that is $a \equiv (b^\perp)^\perp = b$, which was to be proved for (B').

IV. Let us assume that (A') and (B') are satisfied in $\mathcal{L}$. If $a, b \in \mathcal{L}$ and $a \leq b$, then there are two different cases. Firstly, when $b^\perp = 0$, then trivially $b^\perp \leq a^\perp$. Secondly, when $b^\perp \neq 0$, then there exists a $*$ -ultrafilter $\mathcal{R}$ in $\mathcal{L}$ which contains $b^\perp$. But for every such $\mathcal{R}$ we have $a^\perp \in \mathcal{R}$ because $a \in \mathcal{R}$ would contradict $b^\perp \in \mathcal{R}$. Hence by (B') $b^\perp \leq a^\perp$. On the other hand, there is no $*$ -ultrafilter in $\mathcal{L}$ which contains $(0^\perp)^\perp$, so $(0^\perp)^\perp = 0$, and if $a \in \mathcal{L}$, $a \neq 0$, then for every $*$ -ultrafilter $\mathcal{R}$ which contains a we have $a^\perp \notin \mathcal{R}$, $(a^\perp)^\perp \in \mathcal{R}$, i.e. $a \equiv (a^\perp)^\perp$. Contrarily, if $(a^\perp)^\perp \in \mathcal{R}'$ for a $*$ -ultrafilter $\mathcal{R}'$, then $a^\perp \notin \mathcal{R}'$, $a \in \mathcal{R}'$ which imply $(a^\perp)^\perp = a$. Summarizing our results we can state that $\perp$ is an orthocomplementation.

Now we prove the weakly modularity of $\mathcal{L}$. Let $a, b \in \mathcal{L}$, $a \leq b$. Then $a, b \wedge a^\perp \leq b$. Let us assume that $a, b \wedge a^\perp \leq c$, $c \in \mathcal{L}$. To prove $b = a \vee (b \wedge a^\perp)$ it is sufficient to see $b \leq c$. Let us assume that $\mathcal{R}$ is a $*$ -ultrafilter and $b \in \mathcal{R}$. Then there are two different cases: 1. If $a \in \mathcal{R}$, then $c \in \mathcal{R}$, 2. If $a \notin \mathcal{R}$, then $a^\perp \in \mathcal{R}$, $b * a^\perp = (b \vee a) \wedge a^\perp = b \wedge a^\perp \in \mathcal{R}$. By using $b \wedge a^\perp \leq c$ we get $c \in \mathcal{R}$, that is $b \in \mathcal{R}$ implies $c \in \mathcal{R}$, which means that $b \leq c$, which was to be proved. Thus the orthomodularity of $\mathcal{L}$ is proved.

In order to prove (D) it will be sufficient to show that for every $a, b \in \mathcal{L}$, $a \perp b$ there is a complete realization $\mathcal{R}$ of $\mathcal{L}$ such that $a, b \in \mathcal{R}$.

Let $a, b \in \mathcal{L}$, $a \perp b$. Since $a \neq 0$, we have a $*$ -ultrafilter $\mathcal{R}$ in $\mathcal{L}$ containing a . If there is no $*$ -ultrafilter containing a and b , then $b^\perp \in \mathcal{R}$. Hence by (B') we have $a \leq b^\perp$ which is a contradiction. This means that there is a $*$ -ultrafilter $\mathcal{R}'$ which contains a and b , and by Lemma 4. $\mathcal{R}'$ is a complete realization.

V. We assume first that $\mathcal{L}$ is an orthocomplemented lattice and the set $\mathfrak{M}_0$ of two-valued probability measures on $\mathcal{L}$ is order determining. Let $a \in \mathcal{L}$, $a \neq 0$, then there exists an element m of $\mathfrak{M}_0$ satisfying $m(a) = 1$, because $m(a) = 0$ for all $m \in \mathfrak{M}_0$ would imply $a = 0$. So $a \in \mathcal{R} = \{x \in \mathcal{L} | m(x) = 1\}$, where $\mathcal{R}$ is a $*$ -ultrafilter by Lemma 5, which shows that (A') holds in $\mathcal{L}$.

To prove (B') let us suppose now that $a, b \in \mathcal{L}$, $a \neq 0$ and every $*$ -ultrafilter $\mathcal{R}$ containing a contains also b . By Lemma 5. this means that for every $m \in \mathfrak{M}_0$, $m(a) = 1$ implies $m(b) = 1$. Hence $v(a) \leq v(b)$ for all $v \in \mathfrak{M}_0$. Since $\mathfrak{M}_0$ is order determining we obtain $a \leq b$. So (B') also holds in $\mathcal{L}$.

Contrarily, let us suppose that (A') and (B') are satisfied in $\mathcal{L}$. Then $\mathcal{L}$ is an orthomodular and consequently orthocomplemented lattice. Since there exists $*$ -ultrafilters of $\mathcal{L}$ so $\mathfrak{M}_0 \neq \emptyset$. Moreover, if $a, b \in \mathcal{L}$, $a \neq 0$ and $m(a) \leq m(b)$ for all $m \in \mathfrak{M}_0$, then for every $*$ -ultrafilter $\mathcal{R}$ satisfying $a \in \mathcal{R}$ we can define a two-valued probability measure m :

$$m(x) = \begin{cases} 1, & x \in \mathcal{R} \\ 0, & x \in \mathcal{L} \setminus \mathcal{R} \end{cases}$$

which satisfies $m(a) = 1$. Therefore $m(b) = 1$ and $b \in \mathcal{R}$. With the help of (B') we obtain $a \leq b$. So $\mathfrak{M}_0$ is order determining.

VI. Let us suppose that $\mathcal{L}$ satisfies (A), (B) and (C). Denote by $\Omega(\mathcal{L})$ the set of complete realizations of $\mathcal{L}$, and for all $a \in \mathcal{L}$ let

$$X_a = \begin{cases} \text{if } a = 0 \\ \{\mathcal{R} \in \Omega(\mathcal{L}) | a \in \mathcal{R}\} & \text{if } a \neq 0 \end{cases}$$

Let $\mathcal{L}' = \{X_a | a \in \mathcal{L}\}$ and $h(a) = X_a$ for all $a \in \mathcal{L}$. Then $\mathcal{L}'$ is an n -class and $h: \mathcal{L} \rightarrow \mathcal{L}'$ is an isomorphism to $\mathcal{L}'$, because

1. $\Omega(\mathcal{L}) \in \mathcal{L}'$ by $X_1 = \Omega(\mathcal{L})$.
2. If $a \in \mathcal{L}$, then $X_{a^\perp} = \Omega(\mathcal{L}) \setminus X_a$, so $A \in \mathcal{L}' \Rightarrow A' = \Omega \setminus A \in \mathcal{L}'$.
3. Let $a, b \in \mathcal{L}$. Then trivially $a \leq b \Leftrightarrow X_a \subseteq X_b$.

The orthomodularity of $\mathcal{L}$ follows from (C).

VII. (0) $\Rightarrow$ (v) is somewhat trivial, because if $\mathcal{L}$ is isomorphic to an n -class $\mathcal{L}'$, then $\mathcal{L}'$ is orthocomplemented and the set of two-valued probability measures on $\mathcal{L}'$ is order determining, which implies the same properties in $\mathcal{L}$.

This completes the proof of the Theorem.

REMARK. Consider the above defined isomorphism $h: \mathcal{L} \rightarrow \mathcal{L}'$. The following three conditions are equivalent:

- (i) $a \leftrightarrow b$
- (ii) $\sup(h(a), h(b)) = h(a) \cup h(b)$
- (iii) $\inf(h(a), h(b)) = h(a) \cap h(b)$

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I. G. KALMÁR
DEPT. OF MATH.
UNIV. OF DEBRECEN
H-4010 DEBRECEN, PF. 12
HUNGARY

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