

Convergence rates in the Marcinkiewicz strong law of large numbers for Banach space valued random variables with multidimensional indices

By ISTVÁN FAZEKAS (Debrecen)

1. Introduction

Let $(B, |\cdot|)$ be a real separable Banach space and let $\{X_n, n \in N^d\}$ be independent identically distributed (i.i.d.) B -valued random variables (r.v.'s), where N^d denotes the positive integer d -dimensional lattice points (d is a positive integer). For $n, m \in N^d$, $n \leq m$ and $n < m$ are defined coordinatewise and $|n| = \prod_{i=1}^d n_i$ if $n = (n_1, \dots, n_d)$.

Let $S_n = \sum_{k \leq n} X_k$, $n \in N^d$. In [5] the following strong law of large numbers (SLLN) has been proved: if B is of type p ($1 \leq p < 2$), $E|X_1|^p (\log^+ |X_1|)^{d-1} < \infty$ and $EX_1 = 0$, then $S_n/|n|^{1/p} \rightarrow 0$ almost surely (a.s.) as $n \rightarrow \infty$. This is a common generalization of results due to ACOSTA [1] and GUT [7] who have proved this law for $d=1$ and for $d \geq 1$, $B = \mathbf{R}$ (the real numbers) respectively.

The aim of this paper is to give convergence rates in the above SLLN. We shall show that under the assumptions given in this SLLN $\sum_n |n|^{\alpha r - 2} P(|S_n| \geq |n|^{\alpha \varepsilon}) < \infty$ for every $\varepsilon > 0$, where $\alpha r \geq 1$, $\alpha > 1/2$ and $p\alpha > 1$. It will also be proved that $P(|S_n| \geq |n|^{\alpha \varepsilon}) = o(|n|^{1-r\alpha})$ as $n \rightarrow \infty$ if $E|X_1|^r < \infty$, $EX_1 = 0$, $\alpha r \geq 1$, $r \geq 1$ and B is of type p for some $2 \leq p < 1/\alpha$.

These results are well-known for $d=1$ and $B = \mathbf{R}$ (see e.g. [4]). For $d \geq 1$ and $B = \mathbf{R}$ the Marcinkiewicz SLLN and the related convergence rates have been treated by GUT [7]. Convergence rates in the SLLN for B -valued r.v.'s have been obtained by LAI [10] and JAIN [9] if B is arbitrary or B is of type 2 (and $d=1$). AZLAROV and VOLODIN [2] and WOYCZYŃSKI [14] deal with the SLLN and the related convergence rates in B -spaces of type p , and in the case when l_p ($1 \leq p < 2$) is not finitely representable in B respectively. In the case $d=1$ BAKŠTYS and NORVAIŠA [3] have proved more general theorems than our results.

In Section 2 we list some known results that will be used in our proofs. Section 3 deals with convergence rates. We follow the method of GUT [7] and use some ideas of WOYCZYŃSKI [14]. In Section 4 a Chung type SLLN is given.

2. Preliminaries

Let $d(k) = \text{Card} \{ \mathbf{n} : \mathbf{n} \in N^d, |\mathbf{n}| = k \}$ and $M(x) = \sum_{k \leq x} d(k)$. We know that $d(x) = o(x^\sigma)$ as $x \rightarrow \infty$ for every positive σ and $M(x) \sim \text{const. } x(\log^+ x)^{d-1}$ as $x \rightarrow \infty$.

We shall denote by X a B -valued r.v. with the same distribution as X_1 . The following lemma of GUT [7] plays a fundamental role in the proofs of Theorems 3.1 and 3.2.

Lemma 2.1. For $r > 0$ and $m = 0, 1, 2, \dots$ the following statements are equivalent:

$$E|X|^r (\log^+ |X|)^{d-1+m} < \infty;$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 1} (\log |\mathbf{n}|)^m P(|X| \geq |\mathbf{n}|^\alpha) < \infty, \quad \alpha > 0, \quad \varepsilon > 0.$$

Lemma 2.2. Let $0 < r < p \leq 2$ and define $Y_{\mathbf{n}} = X_{\mathbf{n}} \chi_{\{|X_{\mathbf{n}}| \leq \varepsilon |\mathbf{n}|^{1/r}\}}$, where $\chi(A)$ denotes the indicator of the event A . If $E|X|^r (\log^+ |X|)^{d-1+m} < \infty$, then

$$\sum_{\mathbf{n}} (\log |\mathbf{n}|)^m E| |\mathbf{n}|^{-1/r} Y_{\mathbf{n}} |^p < \infty \quad \text{for } m = 0, 1, 2, \dots$$

PROOF. This lemma has been proved in [7] for $p = 2$ and $B = \mathbf{R}$.

$$\begin{aligned} & \sum_{\mathbf{n}} (\log |\mathbf{n}|)^m E| |\mathbf{n}|^{-1/r} Y_{\mathbf{n}} |^p \leq \\ & \leq \sum_{j=1}^{\infty} (\log j)^m j^{-p/r} d(j) \sum_{i=1}^j i^{p/r} P(i-1 < |X|^r \leq i) = \\ & = \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} (\log j)^m j^{-p/r} d(j) \right) i^{p/r} P(i-1 < |X|^r \leq i) \leq \\ & \leq \text{const.} \sum_{i=1}^{\infty} ((\log i)^m i^{-p/r} M(i)) i^{p/r} P(i-1 < |X|^r \leq i) \leq \\ & \leq \text{const.} \sum_{i=1}^{\infty} (\log i)^{m+d-1} i P(i-1 < |X|^r \leq i) < \infty. \end{aligned}$$

In the third step we applied Theorem 1 of [6].

We shall use the Marcinkiewicz—Zygmund inequality which is valid also for B -valued r.v.'s:

Lemma 2.3. (WOYCZYŃSKI [14]). Let B be of type p ($1 \leq p \leq 2$) and $q \geq p$. There exists c such that for any X with $E|X|^q < \infty$, $EX = 0$ the inequality

$$E|S_{\mathbf{n}}|^q \leq c |\mathbf{n}|^{q/p} E|X|^q$$

holds.

Lemma 2.4. (JAIN [9]). Let $\{X_{\mathbf{n}}, \mathbf{n} \in N^d\}$ be i.i.d. symmetric r.v.'s, let j be a positive integer and $t \geq 0$. Then

$$P(|S_{\mathbf{n}}| \geq 3^j t) \leq A_j |\mathbf{n}| P(|X| \geq t) + B_j [P(|S_{\mathbf{n}}| \geq t)]^{2^j},$$

where A_j and B_j are nonnegative constants which depend only on j .

We need the following version of Lévy's inequality:

Lemma 2.5. *Let $\{X_n, n \in N^d\}$ be independent symmetric B -valued r.v.'s. Then*

$$(2.1) \quad P(\max_{j \leq n} |S_j| \geq t) \leq 2^d P(|S_n| \geq t), \quad t \geq 0.$$

PROOF. This can be proved by induction on d using Theorem 2.3 of [8] (which states 2.1 for $d=1$).

The following lemma is a version of Lemma 2 of [3].

Lemma 2.6. *Let $\{Z_n, n \in N^d\}$ be B -valued r.v.'s and let $Z_n^s = Z_n - Z'_n$ be a symmetrization of this sequence. Let ε and δ be positive numbers and suppose that*

$$(2.2) \quad P(|Z_n| < \varepsilon/2 |n|^\alpha) > \delta$$

for $n \prec n_{\varepsilon, \delta}$. Then there exists $n_0 = n_0(\varepsilon, \delta)$ such that

$$(2.3) \quad P(\max_{j \leq n} |Z_j^s| \geq |n|^\alpha \varepsilon/2) \geq \delta P(\max_{j \leq n} |Z_j| \geq \varepsilon |n|^\alpha) \quad \text{for } n \prec n_0.$$

If (2.2) holds for $|n| \geq k_0$, then

$$(2.4) \quad P\left(\sup_{|k| \geq j} \frac{|Z_k^s|}{|k|^\alpha} \geq \frac{\varepsilon}{2}\right) \geq \delta P\left(\sup_{|k| \geq j} \frac{|Z_k|}{|k|^\alpha} \geq \varepsilon\right) \quad \text{for } j \geq k_0.$$

PROOF. Let n_0 be so large that

$$\inf_{j \leq n} P(|Z_j| < \varepsilon/2 |n|^\alpha) > \delta \quad \text{for } n \prec n_0.$$

Since

$$\{|Z_j^s| \geq |n|^\alpha \varepsilon/2\} \supset \{|Z_j| \geq |n|^\alpha \varepsilon\} \cap \{|Z'_j| \leq |n|^\alpha \varepsilon/2\},$$

an application of the "lemma for events" (see [11], p. 246) gives (2.3). The proof of (2.4) is similar.

Remark 2.7. Under the assumptions of the preceding lemma

$$P(|Z_j^s| \geq |n|^\alpha \varepsilon/2) \geq \delta P(|Z_j| \geq |n|^\alpha \varepsilon) \quad \text{for } j \leq n \prec n_{\varepsilon, \delta}.$$

3. Convergence rates

The following results are Banach space analogues of Gut's theorems [7]. The proofs are similar to the proofs given in [7] and will be not given in full detail.

Theorem 3.1. *Let $\{X_n, n \in N^d\}$ be i.i.d. B -valued r.v.'s, let $\alpha r \geq 1$ and $\alpha > 1/2$. Suppose that B is of type p for some $2 \geq p > 1/\alpha$. Let us consider the following statements:*

$$(3.1) \quad E|X|^r (\log^+ |X|)^{d-1} < \infty \quad \text{and if } r \geq 1, \text{ then } EX = 0;$$

$$(3.2) \quad \sum_n |n|^{\alpha r - 2} P(|S_n| \geq |n|^\alpha \varepsilon) < \infty \quad \text{for every } \varepsilon > 0;$$

$$(3.3) \quad \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| \geq |\mathbf{n}|^{\alpha} \varepsilon) < \infty \text{ for every } \varepsilon > 0;$$

$$(3.4) \quad \sum_{j=1}^{\infty} j^{\alpha r - 2} P(\sup_{j \leq |\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} \geq \varepsilon) < \infty \text{ for every } \varepsilon > 0.$$

Then (3.1), (3.2) and (3.3) are equivalent. If $\alpha r > 1$, then all of these statements are equivalent.

PROOF. (a) (3.1) $\Rightarrow$ (3.2). For $r < 1$ this follows from Theorem 4.1 of [7].

If $\alpha r = 1$, then by $p\alpha > 1$, $p > r$. Using Lemma 2.3 (with $q = p$) we obtain

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 2} P(|S_{\mathbf{n}}| \geq 2\varepsilon |\mathbf{n}|^{\alpha}) \leq c\varepsilon^{-p} \sum_{\mathbf{n}} E| |\mathbf{n}|^{-1/r} Y_{\mathbf{n}}|^p + \sum_{\mathbf{n}} P(|X| \geq |\mathbf{n}|^{1/r} \varepsilon) < \infty$$

by Lemmas 2.2 and 2.1.

In the case $\alpha r > 1$ we can suppose that B is of type p for $1 \leq p \leq r$. First we assume that X has symmetric distribution. Lemma 2.4 and Lemma 2.3 (with $q = r$) give

$$\begin{aligned} \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 2} P(|S_{\mathbf{n}}| \geq 3^j |\mathbf{n}|^{\alpha} \varepsilon) &\leq \\ &\leq A_j \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 1} P(|X| \geq |\mathbf{n}|^{\alpha} \varepsilon) + B_j \left(\frac{c}{\varepsilon} E|X|^r \right)^{2^j} \sum_{\mathbf{n}} |\mathbf{n}|^{-\beta}, \end{aligned}$$

where $\beta = 2 - \alpha r + r \left(\alpha - \frac{1}{p} \right) 2^j > 1$ for an appropriate j . Thus Lemma 2.1 implies that the above expression is finite.

If X is not symmetric, then we consider a symmetrization $X^S = X - X'$ of X . According to Theorem 3.2 of [5] $P\left(|S'_{\mathbf{n}}| < \frac{\varepsilon}{2} |\mathbf{n}|^{\alpha}\right) \geq \delta$ for $\mathbf{n} \leq \mathbf{n}_{\varepsilon, \delta}$ (because $\alpha > 1/p$). Since 3.2 holds for the partial sums $S_{\mathbf{n}}^S = S_{\mathbf{n}} - S'_{\mathbf{n}}$ of the symmetrized r.v.'s, by Remark 2.7 (3.2) holds for $S_{\mathbf{n}}$.

(b) (3.2) $\Rightarrow$ (3.4) if $\alpha r > 1$. First we suppose that X is symmetric. In this case we can follow the method used in Proposition 2.1 of [14]. Put $\beta = \alpha r - 2 > -1$.

$$\begin{aligned} \sum_{j=1}^{\infty} j^{\beta} P\left(\sup_{j \leq |\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} \geq \varepsilon 2^{jd}\right) &\leq \text{const.} \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} 2^{id\beta} 2^{id} P\left(\max_{2^{jd} < |\mathbf{k}| \leq 2^{(j+1)d}} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} \geq \varepsilon 2^{jd}\right) \leq \\ &\leq (\text{by (5.2) of [7]}), \end{aligned}$$

$$\begin{aligned} \text{const.} \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} 2^{id\beta} 2^{id} \sum_{|\mathbf{m}|=2^{(j+2)d}} P(|S_{\mathbf{m}}|/|\mathbf{m}|^{\alpha} \geq \varepsilon/2^{jd}) &\leq \\ \leq \text{const.} \sum_{j=0}^{\infty} 2^{d\beta j} 2^{dj} \sum_{|\mathbf{m}|=2^{(j+2)d}} P(|S_{\mathbf{m}}|/|\mathbf{m}|^{\alpha} \geq \varepsilon/2^{jd}). \end{aligned}$$

Grouping terms in blocks $\{\mathbf{k} \in N^d: (2^{l_1}, \dots, 2^{l_d}) \leq \mathbf{k} < (2^{l_1+1}, \dots, 2^{l_d+1})\}$ we obtain that the last expression is not greater than

$$\text{const.} \sum_{\mathbf{m} \in N^d} |\mathbf{m}|^{\beta} P(|S_{\mathbf{m}}| \geq \varepsilon/2^{2jd} |\mathbf{m}|^{\alpha}) < \infty.$$

(c) (3.4) $\Rightarrow$ (3.1). For $d=1$ this implication follows from Theorem 3.3 of Jain [9]. For $d>1$ it can be proved by the method used in [7].

(d) (3.2) $\Rightarrow$ (3.1). First we prove for symmetric r.v.'s. If $\alpha r > 1$, then (3.2) $\Rightarrow$ (3.4) $\Rightarrow$ (3.1). In the case $\alpha r = 1$ a similar computation as in [2] (p. 585) shows that

$$P(|S_n| \geq \varepsilon |n|^{1/r}) \leq 2^{-d} C \sum_{k \leq n} P(|X_k| \geq 2\varepsilon |n|^{1/r})$$

except for finitely many values of n . Thus $\sum_n P(|X| > \varepsilon |n|^{1/r}) < \infty$, and Lemma 2.1 shows that (3.1) holds.

If X is not symmetric, then let X^S be a symmetrization of X . By the symmetrization inequality the sequence S_n^S satisfies (3.2). Thus, by what has already been proved, $E|X^S|^r (\log^+ |X^S|)^{d-1} < \infty$. According to Lemma 2.6 of [9] this expectation is finite for X , too.

(e) We have proved that (3.2) (or (3.3) and (3.4) if $\alpha r > 1$) implies (3.1) without assuming that B is of type p . We also know that (3.2) $\Rightarrow$ (3.4) in the symmetric case. Now we remove the symmetry assumption. From (3.2) there follows (3.1) and by Theorem 3.2 of [5] $P(|S_n| > |n|^\alpha \varepsilon) > \delta$ finitely often (B is of type p). An application of Lemma 2.6 gives the result, because (3.4) holds for S_n^S .

(f) (3.2) $\Rightarrow$ (3.3). In the symmetric case this is a consequence of Lévy's inequality (Lemma 2.5). To remove the symmetry assumption one can argue as in (e).

In the case $\alpha r = 1$ the following result is valid.

Theorem 3.2. Let $\{X_n, n \in N^d\}$ be B -valued i.i.d. r.v.'s. Suppose that B is of type p for some $2 \leq p > r > 0$. Then the following statements are equivalent:

$$(3.5) \quad E|X|^r (\log^+ |X|)^d < \infty \quad \text{and} \quad EX = 0 \quad \text{for} \quad r \geq 1;$$

$$(3.6) \quad \sum_n |n|^{-1} \log |n| P(|S_n| \geq |n|^{1/r} \varepsilon) < \infty \quad \text{for every} \quad \varepsilon > 0;$$

$$(3.7) \quad \sum_n |n|^{-1} \log |n| P(\max_{k \leq n} |S_k| \geq |n|^{1/r} \varepsilon) < \infty \quad \text{for every} \quad \varepsilon > 0;$$

$$(3.8) \quad \sum_{j=1}^{\infty} j^{-1} P(\sup_{j \leq |k|} |S_k|/|k|^{1/r} \geq \varepsilon) < \infty \quad \text{for every} \quad \varepsilon > 0.$$

Proof. (3.5) $\Rightarrow$ (3.6) and in the symmetric case (3.6) $\Rightarrow$ (3.8) can be proved as in Theorem 3.1.

One can prove (3.8) $\Rightarrow$ (3.5) without assuming symmetry and that B is of type p . First we note that Theorem 3.3 in [9] implies that $E|X|^r < \infty$ and $EX=0$ if $r \geq 1$. Using this fact, for $d=1$ the proof is the same as in the real-valued case (Theorem 2 of [4]). For $d>1$ the proof proceeds by induction on d as in [7].

(3.6) $\Rightarrow$ (3.8) in the non-symmetric case and (3.6) $\Rightarrow$ (3.7) follows as in (e) and (f) resp. of the preceding proof.

Theorem 3.3. Let $\{X_n, n \in N^d\}$ be i.i.d. B -valued r.v.'s, $E|X|^r < \infty$ ($\alpha r \geq 1$) and $EX=0$ if $r \geq 1$. If B is of type p for some $2 \leq p > \frac{1}{\alpha}$, then for every positive ε :

$$(3.9) \quad |n|^{\alpha r - 1} P(|S_n| \geq |n|^\alpha \varepsilon) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty;$$

$$(3.10) \quad |n|^{\alpha r - 1} P(\max_{k \leq n} |S_k| \geq |n|^\alpha \varepsilon) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

If $\alpha r > 1$, then

$$(3.11) \quad |\mathbf{n}|^{r\alpha-1} P\left(\sup_{\mathbf{n} \leq \mathbf{k}} \frac{|S_{\mathbf{k}}|}{|\mathbf{k}|^\alpha} \geq \varepsilon\right) \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty.$$

PROOF. (a) If $r\alpha = 1$, then (3.9) is the weak law of large numbers. For $r\alpha > 1$ an application of Lemmas 2.4 and 2.3 proves (3.9) in the symmetric case. If X is not symmetric, then one can use symmetrization and Remark 2.7.

(b) We prove the equivalence of (3.9), (3.10) and (3.11) without the assumption that $E|X|^r < \infty$ and that B is of type p . (3.9) $\Rightarrow$ (3.10) follows from Lévy's inequality by symmetrization.

To prove (3.9) $\Rightarrow$ (3.11) we can assume symmetry. Let $2^{m-1} < \mathbf{n} \leq 2^m$, where $2^m = (2^{m_1}, \dots, 2^{m_d}) \in N^d$ and $\mathbf{1} = (1, \dots, 1) \in N^d$. Then by Lévy's inequality

$$\begin{aligned} |\mathbf{n}|^{r\alpha-1} P\left(\sup_{\mathbf{k} \geq \mathbf{n}} |S_{\mathbf{k}}|/|\mathbf{k}|^\alpha \geq \varepsilon\right) &\leq |2^m|^{r\alpha-1} 2^d \sum_{\mathbf{j} \geq \mathbf{m}} P(|S_{2^j}|/|2^j|^\alpha \geq \varepsilon/2^{d\alpha}) = \\ &= 2^d \sum_{\mathbf{j} \geq \mathbf{m}} |2^m - \mathbf{j}|^{r\alpha-1} \{ |2^j|^{r\alpha-1} P(|S_{2^j}|/|2^j|^\alpha \geq \varepsilon/2^{d\alpha}) \}. \end{aligned}$$

(3.9) implies that $|\mathbf{n}|^{r\alpha-1} P(|S_{\mathbf{n}}| \geq |\mathbf{n}|^\alpha \varepsilon) > \delta$ occurs finitely often, thus the above expression converges to 0 as $\mathbf{n} \rightarrow \infty$.

4. A Chung type SLLN

In this section we deal with a Chung type SLLN for B -valued r.v.'s with multi-dimensional indices. In the case $d=1$ WOYCZYŃSKI [14] proved an SLLN more general than our theorem. In [13] Smythe presented a Chung type SLLN for real random variables with multidimensional indices.

In the proof we shall use the following version of Kolmogorov's inequality and the Three Series Theorem respectively.

Lemma 4.1. Let B be of type p ($1 < p \leq 2$). Let $\{Y_{\mathbf{n}}, \mathbf{n} \in N^d\}$ be independent B -valued r.v.'s with $E|Y_{\mathbf{n}}|^p < \infty$ and $EY_{\mathbf{n}} = 0$ ($\mathbf{n} \in N^d$). Then there exists a $B_{p,d}$ such that

$$P(\max_{\mathbf{k} \leq \mathbf{n}} |Z_{\mathbf{k}}| > \varepsilon) \leq \frac{B_{p,d}}{\varepsilon^p} \sum_{\mathbf{k} \leq \mathbf{n}} E|X_{\mathbf{k}}|^p$$

for every $\varepsilon > 0$, where $Z_{\mathbf{k}} = \sum_{\mathbf{l} \leq \mathbf{k}} X_{\mathbf{l}}$.

PROOF. One can prove this lemma with the help of the Doob—Cairolí inequality (see e.g. [7]).

Lemma 4.2. (In the case $d=1$ see [12]). Besides the assumption of Lemma 4.1 let us suppose that $\sum_{\mathbf{n} \in N^d} E|Y_{\mathbf{n}}|^p < \infty$. Let $Z_{\mathbf{m}}^{\mathbf{n}}$ denote the sum $\sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}}$. Then there exists an event M of zero probability with the following properties:

(a) For every $\varepsilon > 0$ and $\omega \notin M$ one can obtain a $t_0 = t_0(\varepsilon, \omega)$ such that $|Z_{\mathbf{n}}^{\mathbf{m}}(\omega)| < \varepsilon$ if at least one coordinate of $\mathbf{n}$ is greater than t_0 .

(b) For every $\omega \notin M$ there exists $K = K(\omega)$ such that $|Z_{\mathbf{n}}^{\mathbf{m}}(\omega)| \leq K$ for $\mathbf{m} > \mathbf{n}$.

PROOF. An application of Lemma 4.1.

Theorem 4.3. Let B be of type p ($1 < p \leq 2$). Let $\{X_n, n \in N^d\}$ be independent B -valued r.v.'s, $EX_n = 0$. If $\sum_n E|X_n|^p/|n|^p < \infty$, then $S_n/|n| \rightarrow 0$ a.s. as $|n| \rightarrow \infty$.

PROOF. In Lemma 4.2 let $Y_n = X_n/|n|$ ($n \in N^d$). Then

$$\begin{aligned} \frac{|S_m(\omega)|}{|m|} &= \frac{1}{|m|} \left| \sum_{i \leq m} Z_i^m(\omega) \right| \leq \frac{1}{|m|} \sum_{|i| \leq r_0^d} |Z_i^m(\omega)| + \frac{1}{|m|} \sum_{\substack{|i| > r_0^d \\ i \leq m}} |Z_i^m(\omega)| \leq \\ &\leq \frac{r_0^d K(\omega)}{|m|} + \varepsilon < 2\varepsilon \quad \text{if} \quad |m| > \frac{r_0^d K(\omega)}{\varepsilon} \quad (\text{for } \omega \notin M). \end{aligned}$$

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