

On some conditions for the strong law of large numbers

By A. KUCZMASZEWSKA and D. SZYNAL (Lublin)

1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n=0$, $\sigma_n^2=\sigma^2 X_n < \infty$, $n \geq 1$. H. TEICHER [4] has proved that if

$$(i) \quad \sum_k = \sum_{j_k=k}^{\infty} j_k^{-2k} \sigma_{j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} \sigma_{j_{k-1}}^2 \cdots \sum_{j_1=1}^{j_2-1} \sigma_{j_1}^2 < \infty, \quad k \geq 2,$$

$$(ii) \quad n^{-2} \sum_{k=1}^n \sigma_k^2 = o(1) \quad \text{as } n \rightarrow \infty,$$

$$(iii) \quad \sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty$$

for some positive numerical sequence $\{a_n, n \geq 1\}$ with $\sum_{n=1}^{\infty} n^{-4} a_n^2 \sigma_n^2 < \infty$, then $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers, i.e. $S_n/n \rightarrow 0$, a.s., $n \rightarrow \infty$, where $S_n = \sum_{j=1}^n X_j$.

Note that the Kolmogorov series $\sum_1 = \sum_{j=1}^{\infty} j^{-2} \sigma_j^2$ can be considered as the first term of the sequence $\{\sum_k; k \geq 1\}$ where

$$\sum_1 = \sum_{j=1}^{\infty} j^{-2} \sigma_j^2, \quad \sum_k = \sum_{j_k=k}^{\infty} j_k^{-2k} \sigma_{j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} \sigma_{j_{k-1}}^2 \cdots \sum_{j_1=1}^{j_2-1} \sigma_{j_1}^2, \quad k \geq 2.$$

KAI-LAI CHUNG [2] has proved that if a function $\varphi: R \rightarrow R^+$ is nonnegative, even, continuous and nondecreasing on $(0, \infty)$ with $\lim_{x \rightarrow \infty} \varphi(x) = \infty$ such that

(a) $\varphi(x)/x \downarrow$ or (b) $\varphi(x)/x \uparrow$, $\varphi(x)/x^2 \downarrow$ as $x \uparrow \infty$, and if $\{X_k, k \geq 1\}$ is a sequence of independent random variables with $EX_n=0$, $n \geq 1$, and $\sum_{n=1}^{\infty} E\varphi(X_n)/\varphi(n) < \infty$, then $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers.

In [5] there have been given Chung—Teicher type conditions for the strong law of large numbers. Namely, it was shown that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n=0$, $n \geq 1$, and φ is a function satisfying the

condition (a) and

$$(I) \quad \sum_{j_k=k}^{\infty} \frac{E\varphi^2(X_{j_k})}{j_k^{2(k-1)}\varphi^2(j_k)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \frac{E\varphi^2(X_{j_{k-1}})}{\varphi^2(j_{k-1})} \dots$$

$$\dots \sum_{j_1=1}^{j_2-1} j_1^2 \frac{E\varphi^2(X_{j_1})}{\varphi^2(j_1)} < \infty, \quad k \geq 2,$$

$$(II) \quad n^{-2} \sum_{i=1}^n i^2 E\varphi^2(X_i)/\varphi^2(i) = o(1),$$

$$(III) \quad \sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty$$

for some positive numerical sequence $\{a_n, n \geq 1\}$ with

$$(IV) \quad \sum_{n=1}^{\infty} \varphi^2(a_n) E\varphi^2(X_n)/\varphi^4(n) < \infty,$$

or if φ is a function satisfying the condition (b) and

$$(I') \quad \sum_{j_k=k}^{\infty} \frac{E\varphi(X_{j_k})}{j_k^{2(k-1)}\varphi(j_k)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \frac{E\varphi(X_{j_{k-1}})}{\varphi(j_{k-1})} \dots$$

$$\dots \sum_{j_1=1}^{j_2-1} j_1^2 \frac{E\varphi(X_{j_1})}{\varphi(j_1)} < \infty, \quad k \geq 2,$$

$$(II') \quad n^{-2} \sum_{i=1}^n i^2 E\varphi(X_i)/\varphi(i) = o(1),$$

and (III) is satisfied for some numerical sequence $\{a_n, n \geq 1\}$ with

$$(IV') \quad \sum_{n=1}^{\infty} \varphi(a_n) E\varphi(X_n)/\varphi^2(n) < \infty,$$

then $n^{-1} \sum_{k=1}^n (X_k - EX_k I_{[|X_k| < k]}) \rightarrow 0$ a.s., $n \rightarrow \infty$.

Note that for a sequence $\{X_n, n \geq 1\}$ of independent random variables such that

$$P[X_1 = 0] = 1, \quad P\left[X_j = \pm \frac{j}{(\ln \ln j)^{2/(1+\delta)}}\right] = \frac{(\ln \ln j)^2 + 1}{j \ln j}, \quad 0 < \delta < 1$$

$$P[X_j = \pm j^3] = \frac{1}{j^{1+\delta}}, \quad P[X_j = 0] = 1 - \frac{2}{j^{1+\delta}} - \frac{2(\ln \ln j)^2 + 2}{j \ln j},$$

$$j \geq 2 \quad \text{and} \quad \varphi(x) = |x|^{(1+\delta)/2}$$

the series

$$\sum_{j=2}^{\infty} j^{-2} \frac{E\varphi^2(X_j)}{\varphi^2(j)} \sum_{i=1}^{j-1} i^2 \frac{E\varphi^2(X_i)}{\varphi^2(i)}$$

diverges, so the mentioned results do not allow to verify whether that sequence satisfies or not the strong law of large numbers.

The aim of our considerations is to give some new conditions allowing us to verify the strong law of large numbers for larger classes of random variables than the conditions given above.

2. Results

Theorem 1. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$. Let a function $\varphi: R \rightarrow R^+$ be nonnegative, even, continuous and non-decreasing on $(0, \infty)$ with $\lim_{x \rightarrow \infty} \varphi(x) = \infty$ such that

$$(a) \varphi(x)/x \downarrow \text{ or } (b) \varphi(x)/x \uparrow, \varphi(x)/x^2 \downarrow \text{ as } x \uparrow \infty.$$

Suppose that in the case (a)

$$(A) \quad \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi^2(|X_j|)}{\varphi^2(j) + \varphi^2(|X_j|)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi^2(|X_i|)}{\varphi^2(i) + \varphi^2(|X_i|)} < \infty,$$

$$(B) \quad n^{-2} \sum_{i=1}^n i^2 E \frac{\varphi^2(|X_i|)}{\varphi^2(i) + \varphi^2(|X_i|)} = o(1),$$

$$(C) \quad \sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty$$

for some positive numerical sequence $\{a_n, n \geq 1\}$ with

$$(D) \quad \sum_{n=1}^{\infty} \varphi^2(a_n) E \frac{\varphi^2(|X_n|)}{\varphi^4(n) + \varphi^4(|X_n|)} < \infty,$$

or in the case (b)

$$(A_1) \quad \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi(|X_j|)}{\varphi(j) + \varphi(|X_j|)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi(|X_i|)}{\varphi(i) + \varphi(|X_i|)} < \infty,$$

$$(B_1) \quad n^{-2} \sum_{i=1}^n i^2 E \frac{\varphi(|X_i|)}{\varphi(i) + \varphi(|X_i|)} = o(1),$$

and (C) is satisfied for some numerical sequence $\{a_n, n \geq 1\}$ with

$$(D_1) \quad \sum_{n=1}^{\infty} \varphi(a_n) E \frac{\varphi(|X_n|)}{\varphi^2(n) + \varphi^2(|X_n|)} < \infty.$$

Then

$$(1) \quad n^{-1} \sum_{k=1}^n (X_k - EX_k I_{[|X_k| < k]}) \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty.$$

PROOF. Put $X'_n = X_n I_{[|X_n| < n]}$, and let us consider

$$(2) \quad n^{-2} \left[\sum_{i=1}^n (X'_i - EX'_i)^2 \right] = n^{-2} \sum_{i=1}^n (X'_i - EX'_i)^2 + \\ + 2n^{-2} \sum_{j=2}^n (X'_j - EX'_j) \sum_{i=1}^{j-1} (X'_i - EX'_i)$$

Note that in the case (a) $|X'_n|/n \leq \varphi(|X'_n|)/\varphi(n)$ while in the case (b) $|X'_n|^2/n^2 \leq \varphi(|X'_n|)/\varphi(n)$. Taking into account these facts and setting $Y_n = (X'_n - EX'_n)^2 I_{[|X_n| < a_n]}$, $Z_n = Y_n - EY_n$ we get in the case (a)

$$\begin{aligned} \sum_{n=1}^{\infty} E(Z_n/n^2)^2 &\leq \sum_{n=1}^{\infty} E(Y_n^2/n^4) = \sum_{n=1}^{\infty} n^{-4} E(X'_n - EX'_n)^4 I_{[|X_n| < a_n]} \leq \\ &\leq 8 \sum_{n=1}^{\infty} n^{-4} E\{|X'_n|^4 + E^4|X'_n|\} I_{[|X_n| < a_n]} \leq 8 \sum_{n=1}^{\infty} n^{-4} E|X'_n|^4 I_{[|X_n| < a_n]} + \\ &+ 8 \sum_{n=1}^{\infty} n^{-4} E^4|X'_n| = 8 \sum_{n=1}^{\infty} n^{-4} E|X'_n|^4 I_{[|X_n| < a_n]} + \\ &+ 8 \sum_{n=1}^{\infty} n^{-4} E(|X'_n|^4 I_{[|X_n| < a_n]}) + 8 \sum_{n=1}^{\infty} n^{-4} E(|X'_n|^4 I_{[|X_n| \geq a_n]}) \leq \\ &\leq 16 \sum_{n=1}^{\infty} n^{-4} E|X'_n|^4 I_{[|X_n| < a_n]} + 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] \leq \\ &\leq 16 \sum_{n=1}^{\infty} E \frac{\varphi^4(|X_n|) I_{[|X_n| < n]} I_{[|X_n| < a_n]}}{\varphi^4(n)} + 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] \leq \\ &\leq 32 \sum_{n=1}^{\infty} E \frac{\varphi^4(|X_n|)}{\varphi^4(n) + \varphi^4(|X_n|)} I_{[|X_n| < a_n]} + 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] \leq \\ &\leq 32 \sum_{n=1}^{\infty} \varphi^2(a_n) E \frac{\varphi^2(|X_n|)}{\varphi^4(n) + \varphi^4(|X_n|)} + 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty, \end{aligned}$$

and in the case (b)

$$\begin{aligned} \sum_{n=1}^{\infty} E(Z_n/n^2)^2 &\leq 16 \sum_{n=1}^{\infty} n^{-4} E|X'_n|^4 I_{[|X_n| < a_n]} + \\ &+ 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] \leq 16 \sum_{n=1}^{\infty} E \frac{\varphi^2(|X_n|)}{\varphi^2(n)} I_{[|X_n| < a_n]} + \\ &+ 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] \leq 32 \sum_{n=1}^{\infty} E \frac{\varphi^2(|X_n|)}{\varphi^2(n) + \varphi^2(|X_n|)} I_{[|X_n| < a_n]} + \\ &+ 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] \leq 32 \sum_{n=1}^{\infty} \varphi(a_n) E \frac{\varphi(|X_n|)}{\varphi^2(n) + \varphi^2(|X_n|)} + 8 \sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty. \end{aligned}$$

This proves that

$$\sum_{n=1}^{\infty} (Z_n/n^2) < \infty \quad \text{a.s.},$$

which implies by Kronecker's lemma

$$n^{-2} \sum_{i=1}^n (Y_i - EY_i) \rightarrow 0 \quad \text{a.s.}, \quad n \rightarrow \infty.$$

Moreover, we note that in the case (a) the assumption (B) implies

$$\begin{aligned} n^{-2} \sum_{i=1}^n EY_i &= n^{-2} \sum_{i=1}^n E(X'_i - EX'_i)^2 I_{[|X_i| < a_i]} \leq \\ &\leq 4n^{-2} \sum_{i=1}^n E(X_i)^2 I_{[|X_i| < a_i]} \leq 4n^{-2} \sum_{i=1}^n i^2 E(X_i^2/i^2) \leq \\ &\leq 8n^{-2} \sum_{i=1}^n i^2 E \frac{\varphi^2(|X_i|)}{\varphi^2(i) + \varphi^2(|X_i|)} = o(1), \end{aligned}$$

while in the case (b) the assumption (B₁) implies

$$n^{-2} \sum_{i=1}^n EY_i \leq 8n^{-2} \sum_{i=1}^n i^2 E \frac{\varphi(|X_i|)}{\varphi(i) + \varphi(|X_i|)} = o(1).$$

Thus in the case (a) or in the case (b)

$$(3) \quad n^{-2} \sum_{i=1}^n (X'_i - EX'_i)^2 \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \rightarrow \infty.$$

Put now

$$U_{2j} = (X'_j - EX'_j) \sum_{i=1}^{j-1} (X'_i - EX'_i).$$

Then $EU_{2j} = 0$, and by (A) in the case (a)

$$\begin{aligned} &\sum_{j=2}^{\infty} E(U_{2j}/j^2)^2 = \\ &= \sum_{j=2}^{\infty} E j^{-4} [(X'_j - EX'_j) \sum_{i=1}^{j-1} (X'_i - EX'_i)]^2 \leq \sum_{j=2}^{\infty} j^{-2} E \frac{(X'_j)^2}{j^2} \sum_{i=1}^{j-1} i^2 E \frac{(X'_i)^2}{i^2} \leq \\ &\leq \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi^2(|X'_j|)}{\varphi^2(j)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi^2(|X'_i|)}{\varphi^2(i)} \leq \\ &\leq 2^2 \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi^2(|X_j|)}{\varphi^2(j) + \varphi^2(|X_j|)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi^2(|X_i|)}{\varphi^2(i) + \varphi^2(|X_i|)} < \infty \end{aligned}$$

while in the case (b), by (A_1) ,

$$\begin{aligned} \sum_{j=2}^{\infty} E(U_{2j}/j^2)^2 &\leq \sum_{j=2}^{\infty} j^{-2} E \frac{(X'_j)^2}{j^2} \sum_{i=1}^{j-1} i^2 E \frac{(X'_i)^2}{i^2} \leq \\ &\leq 2^2 \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi(|X_j|)}{\varphi(j) + \varphi(|X_j|)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi(|X_i|)}{\varphi(i) + \varphi(|X_i|)} < \infty. \end{aligned}$$

But $\{U_{2j}, j \geq 2\}$ is a martingale difference sequence, so that $\sum_{j=2}^{\infty} U_{2j}/j^2 < \infty$ a.s., which implies

$$(4) \quad n^{-2} \sum_{j=2}^{\infty} U_{2j} = n^{-2} \sum_{j=2}^{\infty} (X'_j - EX'_j) \sum_{i=1}^{j-1} (X'_i - EX'_i) \xrightarrow{\text{a.s.}} 0$$

as $n \rightarrow \infty$. Therefore, by (2)–(4), we obtain

$$n^{-2} \left[\sum_{i=1}^n (X'_i - EX'_i) \right]^2 \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \rightarrow \infty.$$

Thus we have obtained

$$n^{-1} \sum_{i=1}^n (X'_i - EX'_i) \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \rightarrow \infty.$$

Now taking into account the properties of φ and the condition (C), we have

$$\begin{aligned} \sum_{n=1}^{\infty} P[X'_n \neq X_n] &= \sum_{n=1}^{\infty} P[|X_n| \geq n] = \\ &= \sum_{n=1}^{\infty} E \{ I_{[|X_n| \geq n]} I_{[|X_n| \geq a_n]} + I_{[|X_n| < a_n]} I_{[|X_n| \geq n]} \} \leq \\ &\leq \sum_{n=1}^{\infty} P[|X_n| \geq a_n] + 2 \sum_{n=1}^{\infty} E \frac{\varphi^{2r}(|X_n|)}{\varphi^{2r}(n) + \varphi^{2r}(|X_n|)} I_{[|X_n| < a_n]} \leq \\ &\leq \sum_{n=1}^{\infty} P[|X_n| \geq a_n] + 2 \sum_{n=1}^{\infty} \varphi^r(a_n) E \frac{\varphi^r(|X_n|)}{\varphi^{2r}(n) + \varphi^{2r}(|X_n|)} \end{aligned}$$

for any $r \geq 1$. Putting $r=2$ and $r=1$ in the case (a) and (b) respectively, we get

$\sum_{n=1}^{\infty} P[X'_n \neq X_n] < \infty$, which completes the proof of Theorem 1.

Corollary 1. *If (B) and (B_1) are replaced by the condition $n^{-1} \sum_{i=1}^n i E(\varphi(|X_i|)/\varphi(i)) = o(1)$, then $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers, i.e. $S_n/n \rightarrow 0$ a.s. as $n \rightarrow \infty$.*

For a sequence $\{X_n, n \geq 1\}$ of independent random variables having moments only of the first order one can establish the following.

Corollary 2. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$. If

$$\sum_{j=2}^{\infty} j^{-2} E \frac{|X_j|}{j+|X_j|} \sum_{i=1}^{j-1} i^2 E \frac{|X_i|}{i+|X_i|} < \infty, \quad n^{-2} \sum_{i=1}^n i^2 E \frac{|X_i|}{i+|X_i|} = o(1)$$

and $\sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty$ for some positive numerical sequence $\{a_n, n \geq 1\}$ with $\sum_{n=1}^{\infty} a_n E \frac{|X_n|}{n^2 + |X_n|^2} < \infty$ then (1) holds.

To prove the given assertion it is enough to use the case (a) of Theorem 1 with $\varphi(x) = |x|^{1/2}$.

Corollary 2 allows us to state for instance that a sequence $\{X_k, k \geq 1\}$ of independent random variables such that

$$P[X_n = \pm 2^{k/(1+\delta)}/k^{1/(1+\delta)}] = 1/(n^{1+\alpha} 2^{k+1}), \quad k = 1, 2, \dots, \quad 0 < \delta \leq 1, \quad \alpha > 0,$$

$$P[X_n = 0] = 1 - 1/n^{1+\alpha}, \quad n = 1, 2, \dots,$$

satisfies the strong law of large numbers.

To give a generalization of Theorem 1 we need the following lemma [1] p. 329.

Lemma 1. If $\{x_j, 1 \leq j \leq n\}$, are real numbers, $S_n = \sum_{j=1}^n x_j$, and

$$Q_{k,n} = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}, \quad 1 \leq k \leq n,$$

then for $n \geq k \geq 2$, $Q_{k,n} = \sum_{j=k}^n x_j Q_{k-1,j-1}$, and $S_n^k = k! Q_{k,n} + c_k$, where c_k is a generic designation for a finite linear combination (coefficients independent of n) of terms $\prod_{i=1}^m \sum_{j=1}^n x_j^{h_i}$ of order k , that is, $\sum_{i=1}^m h_i = k$, $1 \leq h_i \leq k$, $1 \leq m < k$.

Theorem 2. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$, and φ be a function from Theorem 1. Suppose that in the case (a)

$$(A') \quad \sum_{j_k=k}^{\infty} j_k^{-2(k-1)} E \frac{\varphi^2(|X_{j_k}|)}{\varphi^2(j_k) + \varphi^2(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \cdot \\ \cdot E \frac{\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(j_{k-1}) + \varphi^2(|X_{j_{k-1}}|)} \dots \sum_{j_1=1}^{j_2-1} j_1^2 E \frac{\varphi^2(|X_{j_1}|)}{\varphi^2(j_1) + \varphi^2(|X_{j_1}|)} < \infty,$$

(B), and (C)—(D) are satisfied; or in the case (b)

$$(A_1) \quad \sum_{j_k=k}^{\infty} j_k^{-2(k-1)} E \frac{\varphi(|X_{j_k}|)}{\varphi(j_k) + \varphi(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \cdot \\ \cdot E \frac{\varphi(|X_{j_{k-1}}|)}{\varphi(j_{k-1}) + \varphi(|X_{j_{k-1}}|)} \dots \sum_{j_1=1}^{j_2-1} j_1^2 E \frac{\varphi(|X_{j_1}|)}{\varphi(j_1) + \varphi(|X_{j_1}|)} < \infty,$$

(B₁), and (C)—(D₁) are satisfied. Then (1) holds.

PROOF. Put as previous by $X'_i = X_i I_{[|X_i| < i]}$. Then by Lemma 1, there exist finite constants c_k ; $c_1, c_2, \dots, c_{k-2}$ such that

$$(5) \quad (n^{-1} \sum_{i=1}^n (X'_i - EX'_i))^k = \sum_{h=1}^{k-2} c_h (n^{-1} \sum_{i=1}^n (X'_i - EX'_i))^h A_{k-h,n} + c_k A_{k,n} + k! n^{-k} Q_{k,n},$$

where $0 \leq h \leq k-2$, $A_{k-h,n}$ is a finite linear combination of terms $\prod_{i=1}^m (n^{-h_i} \sum_{j=1}^n (X'_j - EX'_j)^{h_i})$ satisfying $h_i \geq 2$ for $1 \leq i < m < k$, $\sum_{i=1}^m h_i = k-h$, and

$$Q_{0,n} = 1, \quad Q_{k,n} =$$

$$\begin{aligned} &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (X'_{i_1} - EX'_{i_1})(X'_{i_2} - EX'_{i_2}) \dots (X'_{i_k} - EX'_{i_k}) = \sum_{j_k=k}^n (X'_{j_k} - EX'_{j_k}) Q_{k-1,j-1} = \\ &= \sum_{j_k=k}^n (X'_{j_k} - EX'_{j_k}) \sum_{j_{k-1}=k-1}^{j_k-1} (X'_{j_{k-1}} - EX'_{j_{k-1}}) \dots \sum_{j_1=1}^{j_2-1} (X'_{j_1} - EX'_{j_1}). \end{aligned}$$

Note now that for $h_i \geq 2$

$$\left| \prod_{i=1}^m n^{-h_i} \sum_{j=1}^n (X'_j - EX'_j)^{h_i} \right| \leq \prod_{i=1}^m [n^{-2} \sum_{j=1}^n (X'_j - EX'_j)^2]^{h_i/2} \rightarrow 0$$

a.s. as $n \rightarrow \infty$, since $n^{-2} \sum_{i=1}^n (X'_i - EX'_i)^2 \rightarrow 0$ a.s. as $n \rightarrow \infty$, (this fact has been established in the proof of Theorem 1). Thus for $0 \leq h \leq k-2$,

$$(6) \quad A_{k-h,n} \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

Put

$$U_{k,j_k} = (X'_{j_k} - EX'_{j_k}) \sum_{j_{k-1}=k-1}^{j_k-1} (X'_{j_{k-1}} - EX'_{j_{k-1}}) \dots \sum_{j_1=1}^{j_2-1} (X'_{j_1} - EX'_{j_1}).$$

Then $EU_{k,j_k} = 0$, $k=1, 2, \dots$, and moreover, by (A') and the properties of the function φ of the case (a), we get

$$\begin{aligned} &\sum_{j_k=k}^{\infty} E(U_{k,j_k}/j_k^k)^2 \leq \\ &\sum_{j_k=k}^{\infty} j_k^{-2k} E(X'_{j_k} - EX'_{j_k})^2 \dots \sum_{j_{k-1}=k-1}^{j_k-1} E(X'_{j_{k-1}} - EX'_{j_{k-1}})^2 \dots \sum_{j_1=1}^{j_2-1} E(X'_{j_1} - EX'_{j_1})^2 \leq \\ &\leq \sum_{j_k=k}^{\infty} j_k^{-2k} E(X'_{j_k})^2 \sum_{j_{k-1}=k-1}^{j_k-1} E(X'_{j_{k-1}})^2 \dots \sum_{j_1=1}^{j_2-1} E(X'_{j_1})^2 = \\ &= \sum_{j_k=k}^{\infty} j_k^{-2k} E(X'_{j_k})^2 \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \frac{E(X'_{j_{k-1}})^2}{j_{k-1}^2} \dots \sum_{j_1=1}^{j_2-1} j_1^2 \frac{E(X'_{j_1})^2}{j_1^2} \leq \\ &\leq \sum_{j_k=k}^{\infty} j_k^{-2k+1} \frac{E\varphi^2(|X_{j_k}|)}{\varphi^2(j_k)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \frac{E\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(j_{k-1})} \dots \sum_{j_1=1}^{j_2-1} j_1^2 \frac{E\varphi^2(|X_{j_1}|)}{\varphi^2(j_1)} \leq \\ &\leq 2^k \sum_{j_k=k}^{\infty} j_k^{-2k+1} E \frac{\varphi^2(|X_{j_k}|)}{\varphi^2(j_k) + \varphi^2(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 E \frac{\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(j_{k-1}) + \varphi^2(|X_{j_{k-1}}|)} \dots \\ &\quad \dots \sum_{j_1=1}^j j_1^2 E \frac{\varphi^2(|X_{j_1}|)}{\varphi^2(j_1) + \varphi^2(|X_{j_1}|)} < \infty \end{aligned}$$

while in the case (b)

$$\begin{aligned} \sum_{j_k=k}^{\infty} E(U_{k,j_k}/j_k^2)^2 &\leq \sum_{j_k=k}^{\infty} j_k^{-2k+1} \frac{E(X'_{j_k})^2}{j_k^2} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 \frac{E(X'_{j_{k-1}})^2}{j_{k-1}^2} \cdots \sum_{j_1=1}^{j_2-1} j_1^2 \frac{E(X'_{j_1})^2}{j_1^2} \leq \\ &\leq 2^k \sum_{j_k=k}^{\infty} j_k^{-2k+1} E \frac{\varphi(|X_{j_k}|)}{\varphi(j_k) + \varphi(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 E \frac{\varphi(|X_{j_{k-1}}|)}{\varphi(j_{k-1}) + \varphi(|X_{j_{k-1}}|)} \cdots \\ &\quad \cdots \sum_{j_1=1}^{j_2-1} j_1^2 E \frac{\varphi(|X_{j_1}|)}{\varphi(j_1) + \varphi(|X_{j_1}|)} < \infty. \end{aligned}$$

But $\{U_{k,j_k}, k \geq 2\}$ is a martingale difference sequence, so that $\sum_{j_k=k}^{\infty} j_k^{-k} U_{k,j_k} < \infty$ a.s., and by Kronecker's lemma

(7)

$$n^{-k} \sum_{j_k=k}^n U_{k,j_k} = n^{-k} \sum_{j_k=k}^n (X'_{j_k} - EX'_{j_k}) \cdots \sum_{j_1=1}^{j_2-1} (X'_{j_1} - EX'_{j_1}) \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

Thus according to (5) $n^{-1} \sum_{i=1}^n (X'_i - EX'_i)$ is a root of a k -th degree polynomial in which the leading coefficient is unity and the remaining coefficients, by (6) and (7), converge almost surely to zero. Therefore the conclusion

$$n^{-1} \sum_{i=1}^n (X'_i - EX'_i) \rightarrow 0 \text{ a.s., as } n \rightarrow \infty,$$

follows from the well-known relations between the roots and coefficients of a polynomial. But, as in the proof of Theorem 1, for $r=2$ and $r=1$ in the case (a) and (b) respectively

$$\sum_{n=1}^{\infty} P[X'_n \neq X_n] \leq \sum_{n=1}^{\infty} P[|X_n| \geq a_n] + 2 \sum_{n=1}^{\infty} \varphi^r(a_n) E \frac{\varphi^r(|X_n|)}{\varphi^{2r}(n) + \varphi^{2r}(|X_n|)} < \infty$$

which proves (1).

Corollary 3. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$. If

$$\begin{aligned} \sum_{j_k=k}^{\infty} j_k^{-2k+1} E \frac{|X_{j_k}|}{j_k + |X_{j_k}|} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2 E \frac{|X_{j_{k-1}}|}{j_{k-1} + |X_{j_{k-1}}|} \cdots \sum_{j_1=1}^{j_2-1} j_1^2 E \frac{|X_{j_1}|}{j_1 + |X_{j_1}|} < \infty, \\ n^{-2} \sum_{i=1}^n i^2 E \frac{|X_i|}{i + |X_i|} = o(1), \end{aligned}$$

and $\sum_{n=1}^{\infty} P[|X_n| \geq a_n] < \infty$ for some positive numerical sequence $\{a_n, n \geq 1\}$ with

$\sum_{n=1}^{\infty} n^{-2} a_n E(|X_n|/(n^2 + X_n^2)) < \infty$, then (1) holds.

In [3] JEGOROV proved that if $\{b_n, n \geq 1\}$ is a positive, increasing numerical sequence with $\lim_{n \rightarrow \infty} b_n = \infty$ and $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0, n \geq 1$, such that

$$\sum_{j_k=k}^{\infty} b_{j_k}^{-2k} \sigma_{j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} \sigma_{j_{k-1}}^2 \dots \sum_{j_1=1}^{j_k-1} \sigma_{j_1}^2 < \infty \quad \text{and}$$

$\bigwedge_{\varepsilon > 0} \sum_{n=1}^{\infty} P[|X_n| > \varepsilon b_n] < \infty$, then $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers.

The conditions given by Jegorov are more general than Teicher's conditions.

In that way we can generalize Theorem 1 and Theorem 2 from this note. Then we get the following:

Theorem 2'. Let $\{b_n, n \geq 1\}$ be a positive, increasing numerical sequence with $\lim_{n \rightarrow \infty} b_n = \infty$ and $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$ and $\bigwedge_{\varepsilon > 0} \sum_{n=1}^{\infty} P[|X_n| \geq \varepsilon b_n] < \infty$. Suppose that in the case (a)

$$\begin{aligned} \sum_{j_k=k}^{\infty} b_{j_k}^{-2k+1} E \frac{\varphi^2(|X_{j_k}|)}{\varphi^2(b_{j_k}) + \varphi^2(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} b_{j_{k-1}}^2 E \frac{\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(b_{j_{k-1}}) + \varphi^2(|X_{j_{k-1}}|)} \dots \\ \dots \sum_{j_1=1}^{j_k-1} b_{j_1}^2 E \frac{\varphi^2(|X_{j_1}|)}{\varphi^2(b_{j_1}) + \varphi^2(|X_{j_1}|)} < \infty \end{aligned}$$

or in the case (b)

$$\begin{aligned} \sum_{j_k=k}^{\infty} b_{j_k}^{-2k+1} E \frac{\varphi(|X_{j_k}|)}{\varphi(b_{j_k}) + \varphi(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} b_{j_{k-1}}^2 E \frac{\varphi(|X_{j_{k-1}}|)}{\varphi(b_{j_{k-1}}) + \varphi(|X_{j_{k-1}}|)} \dots \\ \dots \sum_{j_1=1}^{j_k-1} b_{j_1}^2 E \frac{\varphi(|X_{j_1}|)}{\varphi(b_{j_1}) + \varphi(|X_{j_1}|)} < \infty. \end{aligned}$$

Then (1) holds.

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ANNA KUCZMASZEWSKA
TECHNICAL UNIVERSITY
UL. DĄBROWSKIEGO 13
20-109 LUBLIN
POLAND

DOMINIK SZYNAL
INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA UNIVERSITY
PL. MARII CURIE-SKŁODOWSKIEJ 1
20-031 LUBLIN
POLAND

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