

On defining the distribution $\delta^{(r)}(f(x))$ for summable f

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In the following we let N be the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers with negligible functions linear sums of the functions n^λ for $\lambda > 0$ and all functions which converge to zero as n tends to infinity.

It follows that if

$$f(n) = f_1(n) + f_2(n),$$

where $f_1(n)$ is a negligible function and the limit as n tends to infinity of $f_2(n)$ exists, then the neutrix limit as n tends to infinity of $f(n)$ exists and

$$N\text{-}\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} f_2(n).$$

Now let ϱ be a fixed infinitely differentiable function having the properties

- (i) $\varrho(x) = 0$ for $|x| \geq 1$,
- (ii) $\varrho(x) \geq 0$,
- (iii) $\varrho(x) = \varrho(-x)$,
- (iv) $\int_{-1}^1 \varrho(x) dx = 1$.

We define the function δ_n by

$$\delta_n(x) = n\varrho(nx)$$

for $n = 1, 2, \dots$

The following definition was given in [2].

Definition 1. Let f be an infinitely differentiable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = (h(x), \varphi(x))$$

for all test functions φ with compact support contained in the interval (a, b) .

The following theorem was proved in [2] and shows that definition 1 is in agreement with the definition of $\delta^{(r)}(f(x))$ given by GELFAND and SHILOV [3].

Theorem 1. Let f be an infinitely differentiable function and suppose that the equation $f(x)=0$ has a single simple root at the point $x=x_1$ in the open interval (a, b) . Then the distribution $\delta^{(r)}(f(x))$ exists and

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_1)|} \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \delta(x-x_1)$$

on the interval (a, b) .

The next definition is an extension of definition 1 and was also given in [2].

Definition 2. Let f be an infinitely differentiable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = (h(x), \varphi(x))$$

for all test functions φ with compact support contained in the interval (a, b) .

The following theorem was then proved.

Theorem 2. Let F be an infinitely differentiable function and suppose that the equation $F(x)=0$ has a single simple root at the point $x=x_1$ in the open interval (a, b) . Then if $f=F^s$, the distribution $\delta^{(r)}(f(x))$ exists and

$$\frac{d}{dx} \delta^{(r)}(f(x)) = f'(x) \delta^{(r+1)}(f(x))$$

on the interval (a, b) for $r=0, 1, 2, \dots$ and $s=1, 2, \dots$. In particular

$$\delta^{(r)}(f(x)) = 0$$

on the interval (a, b) for $r=0, 1, 2, \dots$ and $s=2, 4, \dots$ and

$$(1) \quad \delta^{(r)}((x-x_1)^s) = \frac{r!}{s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1)$$

on the real line for $r=0, 1, 2, \dots$ and $s=1, 3, 5, \dots$.

We now extend definition 2 with the following definition.

Definition 3. Let f be a summable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = (h(x), \varphi(x))$$

for all test functions φ with compact support contained in the interval (a, b) .

Now let f be a summable function. We define the summable functions f_+ and f_- by

$$f_+(x) = \begin{cases} f(x), & x \geq 0, \\ 0, & x < 0 \end{cases}$$

and

$$f_-(x) = \begin{cases} f(x), & x \leq 0, \\ 0, & x > 0. \end{cases}$$

However, in accordance with the usual practice, we define the summable functions x_+^s and x_-^s by

$$x_+^s = \begin{cases} x^s, & x \geq 0, \\ 0, & x < 0 \end{cases} \quad \text{and} \quad x_-^s = \begin{cases} |x|^s, & x \leq 0, \\ 0, & x > 0 \end{cases}$$

for $s=1, 2, \dots$

Theorem 3. Let f be a summable function and suppose that f is continuous and $f(x) \neq 0$ on the closed interval $[a, b]$, where $a < 0 < b$. Then the distribution $\delta^{(r)}(f_+(x-x_1))$ exists and

$$\delta^{(r)}(f_+(x-x_1)) = 0$$

on the interval $(-\infty, b+x_1)$ for $r=0, 1, 2, \dots$. In particular

$$\delta^{(r)}(H(x-x_1)) = 0$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \dots$, where H denotes Heaviside's function.

PROOF. For simplicity of notation we will assume that $x_1=0$. The more general results will follow by translation.

Let φ be an arbitrary test function with compact support contained in the interval $(-\infty, b)$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)) \varphi(x) dx &= \int_{-\infty}^0 \delta_n^{(r)}(0) \varphi(x) dx + \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = \\ &= n^{r+1} \varrho^{(r)}(0) \int_{-\infty}^0 \varphi(x) dx + n^{r+1} \int_0^b \varrho^{(r)}(nf(x)) \varphi(x) dx, \end{aligned}$$

where

$$n^{r+1} \varrho^{(r)}(0) \int_{-\infty}^0 \varphi(x) dx$$

is either negligible or zero. Further, since f is continuous and non-zero on the closed interval $[0, b]$, we can find an integer N such that

$$|nf(x)| \geq 1$$

for $n > N$. It then follows that $\varrho^{(r)}(nf(x))=0$ for $n > N$. Thus

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)) \varphi(x) dx &= \\ &= N\text{-}\lim_{n \rightarrow \infty} n^{r+1} \varrho^{(r)}(0) \int_{-\infty}^0 \varphi(x) dx + \lim_{n \rightarrow \infty} n^{r+1} \int_0^b \varrho^{(r)}(nf(x)) \varphi(x) dx = 0 = (0, \varphi) \end{aligned}$$

and so

$$\delta^{(r)}(f_+(x)) = 0$$

for $r=0, 1, 2, \dots$. This completes the proof of the theorem.

Corollary 3.1. *Let f be a summable function and suppose that f is continuous and $f(x) \neq 0$ on the closed interval $[a, b]$, where $a < 0 < b$. Then the distribution $\delta^{(r)}(f_-(x-x_1))$ exists and*

$$\delta^{(r)}(f_-(x-x_1)) = 0$$

on the interval $(a+x_1, \infty)$ for $r=0, 1, 2, \dots$.

PROOF. Let $g(x) = f(-x)$. Then g is continuous and non-zero for $-b \leq x \leq -a$ and so by the theorem

$$\delta^{(r)}(g_+(x)) = 0$$

on the interval $(-\infty, -a)$ for $r=0, 1, 2, \dots$. Replacing x by $-x$ we see that

$$\delta^{(r)}(g_+(-x)) = \delta^{(r)}(f_-(x)) = 0$$

on the interval (a, ∞) and the result of the corollary follows.

Corollary 3.2. *Let f be a summable function and suppose that f is continuous and $f(x) \neq 0$ on the closed interval $[a, b]$, where $a < 0 < b$. Then the distributions $\delta^{(r)}(f(x-x_1))$ and $\delta^{(r)}(f_+(x-x_1)-f_-(x-x_1))$ exist and $\delta^{(r)}(f(x-x_1)) = \delta^{(r)}(f_+(x-x_1)-f_-(x-x_1)) = 0$ on the interval $(a+x_1, b+x_1)$ for $r=0, 1, 2, \dots$.*

PROOF. Let φ be an arbitrary test function with support contained in the interval (a, b) . Then

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = \int_{-\infty}^0 \delta_n^{(r)}(f(x)) \varphi(x) dx + \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx$$

and

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)-f_-(x)) \varphi(x) dx = \int_{-\infty}^0 \delta_n^{(r)}(-f(x)) \varphi(x) dx + \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx.$$

It was also proved above that

$$\lim_{n \rightarrow \infty} \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = 0$$

and similarly

$$\lim_{n \rightarrow \infty} \int_{-\infty}^0 \delta_n^{(r)}(f(x)) \varphi(x) dx = \lim_{n \rightarrow \infty} \int_{-\infty}^0 \delta_n^{(r)}(-f(x)) \varphi(x) dx = 0.$$

It follows that

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)-f_-(x)) \varphi(x) dx = (0, \varphi)$$

and the result of the corollary follows.

As an *example* consider $f(x) = \cos x$. Then it follows immediately from theorems 1 and 3 that

$$\delta(\cos_+ x) = \sum_{k=0}^{\infty} \delta\left(x - k\pi - \frac{1}{2}\pi\right)$$

and it follows from theorem 1 and corollary 3.2 that

$$\delta(\cos_+ x - \cos_- x) = \sum_{k=0}^{\infty} \delta\left(x - k\pi - \frac{1}{2}\pi\right) + \sum_{k=1}^{\infty} \delta\left(x + k\pi - \frac{1}{2}\pi\right).$$

Further, from theorems 2 and 3, it follows that

$$\delta^{(r)}(\cos_+^2 x) = 0$$

for $r=0, 1, 2, \dots$.

It was proved in [2] that

$$\delta(\sin^3 x) = \sum_{k=-\infty}^{\infty} \frac{1}{6} [\delta(x - k\pi) + \delta''(x - k\pi)]$$

and it follows by translation that

$$\delta(\cos^3 x) = \sum_{k=-\infty}^{\infty} \frac{1}{6} \left[\delta\left(x - k\pi - \frac{1}{2}\pi\right) + \delta''\left(x - k\pi - \frac{1}{2}\pi\right) \right].$$

It now follows from theorem 3 that

$$\delta(\cos_+^3 x) = \sum_{k=0}^{\infty} \frac{1}{6} \left[\delta\left(x - k\pi - \frac{1}{2}\pi\right) + \delta''\left(x - k\pi - \frac{1}{2}\pi\right) \right].$$

All the above results of course hold on the real line.

Theorem 4. Let F be a summable function which is $ms+s+1$ times continuously differentiable on the closed interval $[a, b]$, where $a < 0 < b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x=0$ in the interval $[a, b]$. Then if $f=F^s$, the distribution $\delta^{(r)}(f_+(x-x_1))$ exists on the interval $(-\infty, b+x_1)$ for $r=0, 1, \dots, m$ and $s=1, 2, \dots$. In particular

$$\delta^{(r)}((x-x_1)_+^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \dots$ and $s=1, 2, \dots$.

PROOF. We will again prove the theorem for the case $x_1=0$.

Since $x=0$ is a simple root of the equation $F(x)=0$, it follows that $F'(x) \neq 0$ on the closed interval $[0, c]$, where $0 < c \leq b$. The equation $F(x)=y$ will therefore have an inverse $x=g(y)$ on the interval $[0, c]$ and the function g will be $ms+s+1$ times continuously differentiable.

Now let φ be an arbitrary test function with compact support contained in the interval $(-\infty, c)$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)) \varphi(x) dx &= \int_{-\infty}^0 \delta_n^{(r)}(0) \varphi(x) dx + \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = \\ &= n^{r+1} \varrho^{(r)}(0) \int_{-\infty}^0 \varphi(x) dx + \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx, \end{aligned}$$

where again

$$n^{r+1} \varrho^{(r)}(0) \int_{-\infty}^0 \varphi(x) dx$$

is either negligible or zero. Further, on making the substitution $t^{1/s} = F(x)$ or $x = g(t^{1/s})$ we have

$$\int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = \frac{1}{s} \int_0^{\infty} \delta_n^{(r)}(t) \varphi(g(t^{1/s})) |g'(t^{1/s})| t^{1/s-1} dt.$$

The function

$$\psi(y) = \varphi(g(y)) |g'(y)|$$

is $ms+s$ times continuously differentiable and so since $r \leq m$, we have by Taylor's theorem

$$\psi(y) = \sum_{i=0}^{rs+s-1} \frac{\psi^{(i)}(0)}{i!} y^i + \frac{\psi^{(rs+s)}(\xi y)}{(rs+s)!} y^{rs+s},$$

where $0 \leq \xi \leq 1$. Thus

$$\begin{aligned} s \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx &= \sum_{i=0}^{rs+s-2} \frac{\psi^{(i)}(0)}{i!} \int_0^1 n^r \varrho^{(r)}(u) (u/n)^{(i+1)/s-1} du + \\ &+ \frac{\psi^{(rs+s-1)}(0)}{(rs+s-1)!} \int_0^1 \varrho^{(r)}(u) u^r du + \int_0^1 \frac{\psi^{(rs+s)}(\xi(u/n)^{1/s})}{(rs+s)!} n^r \varrho^{(r)}(u) (u/n)^{r+1/s} du, \end{aligned}$$

where the substitution $nt=u$ has been made. It follows that

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx &= \frac{\psi^{(rs+s-1)}(0)}{s(rs+s-1)!} \int_0^1 \varrho^{(r)}(u) u^r du = \\ &= \frac{(-1)^r \psi^{(rs+s-1)}(0)}{2s(rs+s-1)!} = N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)) \varphi(x) dx. \end{aligned}$$

This proves the existence of $\delta^{(r)}(f_+(x))$ on the interval $(-\infty, c)$ for $r=0, 1, \dots, m$ and $s=1, 2, \dots$. We of course have $\delta^{(r)}(f_+(x))=0$ on the interval $(1/2 c, b)$ by corollary 3.2.

In the particular case when $F=x$, F is infinitely differentiable, $f_+=x^s$ and ψ is identical to φ . It follows that

$$(\delta^{(r)}(x_+^s), \varphi(x)) = \frac{(-1)^r r! \varphi^{(rs+s-1)}(0)}{2s(rs+s-1)!} = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} (\delta^{(rs+s-1)}(x), \varphi(x))$$

and so

$$\delta^{(r)}(x_+^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x)$$

for $r=0, 1, 2, \dots$ and $s=1, 2, \dots$. This completes the proof of the theorem.

Corollary 4.1. Let F be a summable function which is $ms+s+1$ times continuously differentiable on the closed interval $[a, b]$, where $a < 0 < b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x=0$ in the interval $[a, b]$. Then if $f=F^s$, the distribution $\delta^{(r)}(f_-(x-x_1))$ exists on the interval $(a+x_1, \infty)$ for $r=0, 1, \dots, m$ and $s=1, 2, \dots$. In particular

$$\delta^{(r)}((x-x_1)_-^s) = \frac{(-1)^r r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \dots$ and $s=1, 2, \dots$.

PROOF. We note that

$$\delta^{(r)}(f_-(x)) = \delta^{(r)}(f_+(-x)),$$

the right hand side existing by the theorem.

In particular we have

$$\delta^{(r)}(x_+^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x)$$

and replacing x by $-x$ we have

$$\delta^{(r)}(x_-^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(-x) = \frac{(-1)^r r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x).$$

Corollary 4.2. Let F be a summable function which is $ms+s+1$ times continuously differentiable on the closed interval $[a, b]$, where $a < 0 < b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x=0$ in the interval $[a, b]$. Then if $f=F^s$, the distributions $\delta^{(r)}(f(x))$ and $\delta^{(r)}(f_+(x)-f_-(x))$ exist on the interval $(a+x_1, b+x_1)$ for $r=0, 1, \dots, m$ and $s=1, 2, \dots$. In particular

$$(2) \quad \delta^{(r)}(\operatorname{sgn}(x-x_1)(x-x_1)^s) = 0$$

on the interval $(-\infty, \infty)$ for $r=0, 2, 4, \dots$ and $s=2, 4, \dots$

$$(3) \quad \delta^{(r)}(|x-x_1|^s) = 0$$

on the interval $(-\infty, \infty)$ for $r, s=1, 3, \dots$,

$$(4) \quad \delta^{(r)}(\operatorname{sgn}(x-x_1)(x-x_1)^s) = \frac{r!}{s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=1, 3, \dots$ and $s=2, 4, \dots$ and

$$(5) \quad \delta^{(r)}(|x-x_1|^s) = \frac{r!}{s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=0, 2, 4, \dots$ and $s=1, 3, \dots$.

PROOF. We have

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx &= \int_{-\infty}^0 \delta_n^{(r)}(f(x)) \varphi(x) dx + \int_0^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = \\ &= \int_{-\infty}^{\infty} \delta_n^{(r)}(f_-(x)) \varphi(x) dx + \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)) \varphi(x) dx - \int_{-\infty}^{\infty} \delta_n^{(r)}(0) \varphi(x) dx, \end{aligned}$$

the last term being either negligible or zero. It follows that

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = (\delta^{(r)}(f_+(x)) + \delta^{(r)}(f_-(x)), \varphi(x)),$$

so that in fact

$$\delta^{(r)}(f(x)) = \delta^{(r)}(f_+(x)) + \delta^{(r)}(f_-(x)).$$

Similarly we can prove that

$$\begin{aligned} \delta^{(r)}(f_+(x) - f_-(x)) &= \delta^{(r)}(f_+(x)) + \delta^{(r)}(-f_-(x)) = \\ &= \delta^{(r)}(f_+(x)) + (-1)^r \delta^{(r)}(f_-(x)). \end{aligned}$$

Equations (2), (3), (4) and (5) now follow from these results on using theorem 4 and corollary 4.1.

We note that the following more general results hold.

$$\begin{aligned} \delta^{(r)}((x-x_1)_+^s + (x-x_1)_-^m) &= \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1) + \\ &+ \frac{(-1)^r r!}{2m(rm+m-1)!} \delta^{(rm+m-1)}(x-x_1), \\ \delta^{(r)}((x-x_1)_+^s - (x-x_1)_-^m) &= \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!} \delta^{(rs+s-1)}(x-x_1) + \\ &+ \frac{r!}{2m(rm+m-1)!} \delta^{(rm+m-1)}(x-x_1) \end{aligned}$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \dots$ and $s, m=1, 2, \dots$. Equations (1), (2), (3), (4) and (5) are of course particular cases of these results.

As a final example consider $F(x)=\sin x$ and $f(x)=\sin^2 x$. It follows from theorem 2 that

$$\delta^{(r)}(\sin^2 x) = 0$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \dots$. Further, for arbitrary test function φ

$$\psi(y) = \varphi(\sin^{-1} y)(1-y^2)^{-1/2}$$

and it can be shown that

$$\psi'(0) = \varphi'(0),$$

$$\psi'''(0) = 4\varphi'(0) + \varphi'''(0).$$

It follows from theorem 4 and its corollaries that

$$(\delta(\sin_+^2 x), \varphi(x)) = \frac{1}{4} \psi'(0) = \frac{1}{4} \varphi'(0),$$

$$(\delta'(\sin_+^2 x), \varphi(x)) = -\frac{1}{24} \psi'''(0) = -\frac{1}{6} \varphi'(0) - \frac{1}{24} \varphi'''(0)$$

and so

$$\delta(\sin_+^2 x) = -\frac{1}{4} \delta'(x),$$

$$\delta(\sin_-^2 x) = \frac{1}{4} \delta'(x),$$

$$\delta(\operatorname{sgn} x \cdot \sin^2 x) = -\frac{1}{2} \delta'(x),$$

$$\delta'(\sin_+^2 x) = \frac{1}{6} \delta'(x) + \frac{1}{24} \delta'''(x),$$

$$\delta'(\sin_-^2 x) = -\frac{1}{6} \delta'(x) - \frac{1}{24} \delta'''(x).$$

$$\delta'(\operatorname{sgn} x \cdot \sin^2 x) = \frac{1}{3} \delta'(x) + \frac{1}{12} \delta'''(x)$$

on the interval $(-\infty, \infty)$.

References

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