

The fundamental equation of information on open domain

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Introduction

The fundamental equation of information

$$(1) \quad f(x) + (1-x)f\left(\frac{y}{1-x}\right) = f(y) + (1-y)f\left(\frac{x}{1-y}\right)$$

receives much attention in the study of measures of information, and is the subject of Chapter 3 of the book [1] by ACZÉL and DARÓCZY. The general solution is known when it is supposed to hold for all $x, y \in]0, 1[$ with $x+y \leq 1$. In the study of information measures of higher dimensions Aczél observed that it is desirable to know the general solution of (1) when it is supposed to hold in a smaller open domain

$$D^0 = \{(x, y) \mid x, y \in]0, 1[\text{ with } x+y < 1\}.$$

We shall solve (1) on D^0 in the next section. While the deduction will become more elaborate the essential steps are in line with those reported in [1].

The general solution

Theorem. *A function $f:]0, 1[\rightarrow R$ satisfies the functional equation (1) on the open domain D^0 if, and only if, it is of the form*

$$(2) \quad f(x) = \varphi(x) + \varphi(1-x) + ax \quad \text{for all } x \in]0, 1[,$$

where a is an arbitrary constant and $\varphi:]0, \infty[\rightarrow R$ is a function satisfying

$$(3) \quad \varphi(uv) = u\varphi(v) + v\varphi(u) \quad \text{for all } u, v > 0.$$

PROOF. Let f be a solution of (1) and define $F:]0, \infty[^2 \rightarrow R$ by

$$(4) \quad F(u, v) = (u+v)f\left(\frac{v}{u+v}\right) \quad \text{for all } u, v > 0.$$

Obviously F satisfies the homogeneity

$$(5) \quad F(tu, tv) = tF(u, v) \quad \text{for all } t, u, v > 0.$$

We show that F satisfies

$$(6) \quad F(u+v, w) + F(u, v) = F(u+w, v) + F(u, w) \quad \text{for all } u, v, w > 0$$

by the following computations using (1):

$$\begin{aligned} F(u+v, w) + F(u, v) &= (u+v+w)f\left(\frac{w}{u+v+w}\right) + (u+v)f\left(\frac{v}{u+v}\right) = \\ &= (u+v+w)\left[f\left(\frac{w}{u+v+w}\right) + \left(1 - \frac{w}{u+v+w}\right)f\left(\frac{v/(u+v+w)}{1 - (w/(u+v+w))}\right)\right] = \\ &= (u+v+w)\left[f\left(\frac{v}{u+v+w}\right) + \left(1 - \frac{v}{u+v+w}\right)f\left(\frac{w/(u+v+w)}{1 - (v/(u+v+w))}\right)\right] = \\ &= (u+w+v)f\left(\frac{v}{u+w+v}\right) + (u+w)f\left(\frac{w}{u+w}\right) = F(u+w, v) + F(u, w). \end{aligned}$$

Now we define $h:]0, \infty[\rightarrow \mathbb{R}$ and $G:]0, \infty[^2 \rightarrow \mathbb{R}$ by

$$(7) \quad h(u) = F(u, 1) - F(1, u) \quad \text{for all } u > 0,$$

and

$$(8) \quad G(u, v) = F(u, v) + h(v) \quad \text{for all } u, v > 0.$$

Evidently h satisfies

$$(9) \quad h(1) = 0.$$

We claim that G satisfies the following equations:

$$(10) \quad G(u, v) = G(v, u) \quad \text{for all } u, v > 0,$$

$$(11) \quad G(u+v, w) + G(u, v) = G(u+w, v) + G(u, w) \quad \text{for all } u, v, w > 0.$$

To support the above, we fix $w=1$ in (6) and then subtract from it the equation obtained by interchanging u and v to get

$$F(u, v) - F(v, u) = F(u+1, v) + F(u, 1) - F(v+1, u) - F(v, 1).$$

On the other hand using (6) we have $F(u+1, v) - F(v+1, u) = F(1, v) - F(1, u)$, and so the above equation leads to $F(u, v) - F(v, u) = F(1, v) - F(1, u) + F(u, 1) - F(v, 1)$, proving (10). Adding $h(v) + h(w)$ to the two sides of (6) we get (11). Next, we proceed to show that the function $H:]0, \infty[^2 \rightarrow \mathbb{R}$ defined by

$$(12) \quad H(u, v) = G(u, v) + h(u+v) - h(u) - h(v) \quad \text{for all } u, v > 0$$

satisfies

$$(13) \quad H(u, v) = H(v, u) \quad \text{for all } u, v > 0,$$

$$(14) \quad H(u+v, w) + H(u, v) = H(u+w, v) + H(u, w) \quad \text{for all } u, v, w > 0,$$

and

$$(15) \quad H(tu, tv) = tH(u, v) \quad \text{for all } t, u, v > 0.$$

Equations (13) and (14) follow easily from (10) and (11). In order to get (15), we

first observed that since F satisfies the homogeneity (5) and so

$$G(tu, tv) - tG(u, v) = F(tu, tv) + h(tv) - tF(u, v) - th(v) = h(tv) - th(v).$$

Since the left hand side is symmetric in u and v , we get $h(tv) - th(v) = h(tu) - th(u)$. By fixing $u=1$ and using (9) we get

$$(16) \quad h(tv) - th(v) = h(t) \quad \text{for all } t, v > 0.$$

Thus $G(tu, tv) - tG(u, v) = h(t)$. But then we have

$$\begin{aligned} H(tu, tv) - tH(u, v) &= \\ &= G(tu, tv) + h(tu + tv) - h(tu) - h(tv) - t[G(u, v) + h(u + v) - h(u) - h(v)] = \\ &= h(t) + [h(tu + tv) - th(u + v)] - [h(tu) - th(u)] - [h(tv) - th(v)] = \\ &= h(t) + h(t) - h(t) - h(t) = 0, \end{aligned}$$

proving (15). By a result of JESSEN, KARPf and THORUP [2] we know that H satisfies (13), (14) and (15) if, and only if, it is of the form

$$(17) \quad H(u, v) = \varphi(u) + \varphi(v) - \varphi(u + v) \quad \text{for all } u, v > 0,$$

where φ is a solution of (3).

To obtain the explicit form of h , we solve (16) as follows: Using the symmetry of $h(tv)$ in t and v , we get from (16) $h(t) + th(v) = h(v) + vh(t)$. By fixing in it $t=2$ this implies

$$(18) \quad h(v) = -av + a \quad \text{for all } v > 0,$$

where $a = -h(2)$ is a constant. With these known forms of H and h we can determine f , F and G . In fact,

$$\begin{aligned} f(x) &= F(1-x, x) = G(1-x, x) - h(x) = \\ &= H(1-x, x) - h(1) + h(1-x) + h(x) - h(x) = \varphi(1-x) + \varphi(x) - \varphi(1) + h(1-x) = \\ &= \varphi(1-x) + \varphi(x) + ax \end{aligned}$$

which is the asserted from (2).

The converse is straightforward.

References

- [1] J. ACZÉL and Z. DARÓCZY, On Measures of Information and their Characterizations (*Academic Press, New York*, 1975).
- [2] B. JESSEN, J. KARPf and A. THORUP, Some Functional Equations in Groups and Rings, *Math. Scand.* 22 (1968), 257—265.

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