

## Generalized-homogeneous deviation means

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**1. Introduction.** Let  $\varphi: \mathbf{R}_+ \rightarrow \mathbf{R}$  be a strictly monotone and continuous function. For the quasiarithmetic mean  $M_{n,\varphi}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) defined by

$$(1.1) \quad M_{n,\varphi}(\underline{x}) := \varphi^{-1} \left[ \frac{1}{n} \sum_{i=1}^n \varphi(x_i) \right]$$

whenever  $\underline{x} = (x_1, \dots, x_n) \in \mathbf{R}_+^n$ ,  $n \in \mathbf{N}$  the following basic result is well known (see HARDY—LITTLEWOOD—PÓLYA [8, Theorem 84.]): *If (1.1) is homogeneous i.e.  $M_{n,\varphi}(t\underline{x}) = tM_{n,\varphi}(\underline{x})$  for  $t \in \mathbf{R}_+$ ,  $\underline{x} \in \mathbf{R}_+^n$ ,  $n \in \mathbf{N}$  then there exists  $a \in \mathbf{R}$ , such that*

$$(1.2) \quad M_{n,\varphi}(\underline{x}) = M_{n,a}(\underline{x})$$

for each  $\underline{x} \in \mathbf{R}_+^n$  and  $n \in \mathbf{N}$  where

$$(1.3) \quad M_{n,a}(\underline{x}) := \begin{cases} \left[ \frac{1}{n} \sum_{i=1}^n x_i^a \right]^{1/a}, & \text{if } a \neq 0, \\ \left[ \prod_{i=1}^n x_i \right]^{1/n}, & \text{if } a = 0. \end{cases}$$

One of the known generalizations of quasiarithmetic means is the concept of *deviation means* (see DARÓCZY [3], [4]).

The function  $E: \mathbf{R}_+^2 \rightarrow \mathbf{R}$  is called a *deviation* on  $\mathbf{R}_+$  if the following properties are satisfied:

(E1) The function  $y \rightarrow E(x, y)$  is strictly monotone decreasing and continuous for any fixed  $x \in \mathbf{R}_+$ .

(E2)  $E(x, x) = 0$  for any  $x \in \mathbf{R}$ .

Denote by  $\varepsilon(\mathbf{R}_+)$  the set of all deviations defined on  $\mathbf{R}_+$ . It is known (see DARÓCZY [3]) that, for arbitrary  $n \in \mathbf{N}$  and  $\underline{x} = (x_1, \dots, x_n) \in \mathbf{R}_+^n$ , the equation

$$(1.4) \quad \sum_{i=1}^n E(x_i, y) = 0$$

has a unique solution  $y_0 \in \mathbf{R}_+$  such that  $\min \{x_i | 1 \leq i \leq n\} \leq y_0 \leq \max \{x_i | 1 \leq i \leq n\}$  is also satisfied. Therefore the symmetrical value  $y_0 =: \mathfrak{M}_{n,E}(\underline{x})$  is called the *E-deviation mean* of  $\underline{x}$ .

If  $\varphi: \mathbf{R}_+ \rightarrow \mathbf{R}$  is a strictly monotone and continuous function then

$$(1.5) \quad E_\varphi(x, y) := \varepsilon[\varphi(x) - \varphi(y)], \quad (x, y \in \mathbf{R}_+)$$

is a deviation on  $\mathbf{R}_+$ , where  $\varepsilon=1$  if  $\varphi$  increases and  $\varepsilon=-1$  if  $\varphi$  decreases. It is easy to see that

$$(1.6) \quad \mathfrak{M}_{n,E,\varphi}(\underline{x}) = M_{n,\varphi}(\underline{x})$$

i.e. quasiarithmetic means are deviation means, too. Therefore, the following question seems to be natural: What can we state about homogeneous deviation means? The following result answers this question (DARÓCZY [3]): *If  $E \in \varepsilon(\mathbf{R}_+)$  and  $\mathfrak{M}_{n,E}(t\underline{x}) = t\mathfrak{M}_{n,E}(\underline{x})$  for any  $t \in \mathbf{R}_+$ ,  $\underline{x} \in \mathbf{R}_+^n$ ,  $n \in \mathbf{N}$ , then there exists  $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  with  $h(1)=1$  and  $f: \mathbf{R}_+ \rightarrow \mathbf{R}$  with  $\operatorname{sgn} f(x) = \operatorname{sgn}(1-x)$ , ( $x \in \mathbf{R}_+$ ) such that*

$$(1.7) \quad E(x, y) = h(y) f\left(\frac{x}{y}\right).$$

This result is unsatisfactory from many points of view. On the one hand it is not reversible since (1.7) is not a deviation for any given  $h$  and  $f$ , on the other hand there still exist too many homogeneous deviation means. These problems lead us to introduce a new homogeneity property.

*Definition.* Let  $E \in \varepsilon(\mathbf{R}_+)$ . The deviation mean  $\mathfrak{M}_{n,E}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) is said to be  $k$ -homogeneous if

$$(1.8) \quad \mathfrak{M}_{nk,E}(t \circ \underline{x}) = \mathfrak{M}_{k,E}(t) \mathfrak{M}_{n,E}(\underline{x})$$

for any  $t = (t_1, \dots, t_k) \in \mathbf{R}_+^k$  and  $\underline{x} \in \mathbf{R}_+^n$ ,  $n \in \mathbf{N}$  and where

$$(1.9) \quad t \circ \underline{x} = (t_1 x_1, \dots, t_1 x_n, \dots, t_k x_1, \dots, t_k x_n) \in \mathbf{R}_+^{kn}.$$

We remark that  $k$ -homogeneous ( $k \geq 2$ ) deviation means are also 1-homogeneous means i.e. homogeneous means since substituting  $t = (t, \dots, t)$  ( $t \in \mathbf{R}_+$ ) in (1.8) we obtain

$$\mathfrak{M}_{k,E}(t) = t \quad \text{and} \quad \mathfrak{M}_{kn,E}(t \circ \underline{x}) = \mathfrak{M}_{n,E}(t\underline{x}).$$

On the other hand it can easily be seen that homogeneous quasiarithmetic means are  $k$ -homogeneous means, too, since

$$(1.10) \quad \begin{aligned} M_{kn,a}(t \circ \underline{x}) &= \left( \frac{1}{kn} \sum_{j=1}^k \sum_{i=1}^n (t_j x_i)^a \right)^{1/a} = \\ &= \left( \frac{1}{k} \sum_{j=1}^k t_j^a \frac{1}{n} \sum_{i=1}^n x_i^a \right)^{1/a} = M_{k,a}(t) M_{n,a}(\underline{x}) \end{aligned}$$

if  $a \neq 0$ . Either by letting  $a \rightarrow 0$  in (1.10) or by a direct computation we obtain that (1.10) is valid for  $a=0$  as well.

Deviation means which are  $k$ -homogeneous for any  $k$  are called *multiplicative* means. The class of these means is known (DARÓCZY—PÁLES [7, Theorem 9]): *If  $E \in \varepsilon(\mathbf{R}_+)$  and  $\mathfrak{M}_{n,E}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) is multiplicative then there exist a multiplicative function  $m: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  (i.e.  $m(xy) = m(x)m(y)$  if  $x, y \in \mathbf{R}_+$ ) and a constant  $a \in \mathbf{R} \setminus \{0\}$  such that either*

$$(1.11) \quad \mathfrak{M}_{n,E}(\underline{x}) = \exp \left[ \frac{\sum_{i=1}^n m(x_i) \ln x_i}{\sum_{i=1}^n m(x_i)} \right], \quad (\underline{x} \in \mathbf{R}_+^n)$$

or

$$(1.12) \quad \mathfrak{M}_{n,E}(\underline{x}) = \left[ \frac{\sum_{i=1}^n m(x_i) x_i^a}{\sum_{i=1}^n m(x_i)} \right]^{1/a}, \quad (\underline{x} \in \mathbf{R}_+^n).$$

Conversely, the means standing on the right hand side of (1.11) and (1.12) are multiplicative deviation means.

In the present article we shall investigate the  $k$ -homogeneous deviation means for some fixed  $k \geq 2$ . We shall prove a surprising result:

If  $E$  is differentiable with respect to its second variable and its derivative is non-vanishing then  $\mathfrak{M}_{n,E}$  ( $n \in \mathbf{N}$ ) is a  $k$ -homogeneous deviation mean if and only if there exist a multiplicative function  $m$  and a constant  $a \in \mathbf{R} \setminus \{0\}$  such that either (1.11) or (1.12) is satisfied.

With the help of this result we can easily see that the  $k$ -homogeneity of a regular deviation mean (for some fixed  $k \geq 2$ ) implies the multiplicativity of this mean.

**2. Basic functional equation.** In this section we deduce a functional equation which plays an essential role in our discussion.

**Theorem 1.** Let  $E \in \varepsilon(\mathbf{R}_+)$  and assume that  $\mathfrak{M}_{n,E}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) is a  $k$ -homogeneous mean for some fixed  $k \geq 2$ . Then the functions

$$(2.1) \quad f(x) := E(x, 1), \quad \mu(x) := \mathfrak{M}_{k,E}(x, 1, \dots, 1), \quad x \in \mathbf{R}_+$$

satisfy the functional equation

$$(2.2) \quad \frac{f(xy)}{f(x)f(y)} + \frac{k-1}{f(x)} = \frac{f(\mu(x)y)}{f(\mu(x))f(y)}$$

for any  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .

**PROOF.** By the definition of deviation means,  $\mathfrak{M}_{n,E}$  is  $k$ -homogeneous if and only if

$$(2.3) \quad \sum_{i=1}^n \sum_{j=1}^k E(t_j x_i, \mathfrak{M}_{k,E}(t) \mathfrak{M}_{n,E}(\underline{x})) = 0$$

for any  $\underline{t} = (t_1, \dots, t_k) \in \mathbf{R}_+^k$ ,  $\underline{x} = (x_1, \dots, x_n) \in \mathbf{R}_+^n$ ,  $n \in \mathbf{N}$ . Let  $\underline{t} \in \mathbf{R}_+^k$  be fixed and define  $F_{\underline{t}}: \mathbf{R}_+^2 \rightarrow \mathbf{R}$  by

$$(2.4) \quad F_{\underline{t}}(x, y) := \sum_{j=1}^k E(t_j x, \mathfrak{M}_{k,E}(\underline{t}) y).$$

Then  $F_{\underline{t}} \in \varepsilon(\mathbf{R}_+)$  since it is strictly monotone decreasing and continuous in the second variable, further, by the homogeneity of  $\mathfrak{M}_{n,E}$ ,

$$F_{\underline{t}}(x, x) = \sum_{j=1}^k E(t_j x, \mathfrak{M}_{k,E}(\underline{t}) x) = \sum_{j=1}^k E(t_j x, \mathfrak{M}_{k,E}(x \underline{t})) = 0.$$

Applying (2.4) the equation (2.3) turns into

$$\sum_{i=1}^n F_t(x_i, \mathfrak{M}_{n,E}(x)) = 0,$$

from which we obtain

$$(2.5) \quad \mathfrak{M}_{n,E}(x) = \mathfrak{M}_{n,F_t}(x)$$

for any  $x \in \mathbb{R}_+^n$ ,  $n \in \mathbb{N}$ . It is known (see DARÓCZY—PÁLES [6, Theorem 1]) that (2.5) is satisfied if and only if

$$(2.6) \quad F_t(u, v)E(w, v) = F_t(w, v)E(u, v)$$

for any  $0 < u \leq v \leq w$ .

We see immediately that (2.6) is valid not only in the region  $0 < u \leq v \leq w$  but for any  $0 < u, v, w$ . E.g. let  $0 < u \leq v$  and  $0 < w \leq v$ . Choose  $w^* > v$  and apply (2.6) for the values  $0 < u \leq v < w^*$  and  $0 < w \leq v < w^*$ . Then we have

$$(2.7) \quad F_t(u, v)E(w^*, v) = F_t(w^*, v)E(u, v),$$

$$(2.8) \quad F_t(w^*, v)E(w, v) = F_t(w, v)E(w^*, v).$$

Multiplying (2.7) by (2.8) and dividing the resulting equation by

$$E(w^*, v)F_t(w^*, v) > 0$$

we obtain (2.6) just for  $0 < u, w \leq v$ . In the case  $0 < v \leq u, w$  the proof of (2.6) is completely similar.

Now let  $x, y \in \mathbb{R}_+ \setminus \{1\}$  be arbitrary and substitute into (2.6)

$$u = y\mu(x), \quad v = 1, \quad w = \mu(x),$$

$$t = \left( \frac{x}{\mu(x)}, \frac{1}{\mu(x)}, \dots, \frac{1}{\mu(x)} \right) \in \mathbb{R}_+^k.$$

Then, since  $\mathfrak{M}_{k,E}$  is a homogeneous mean,

$$\mathfrak{M}_{k,E}(t) = \mathfrak{M}_{k,E}\left(\frac{x}{\mu(x)}, \frac{1}{\mu(x)}, \dots, \frac{1}{\mu(x)}\right) = \frac{1}{\mu(x)} \mathfrak{M}_{k,E}(x, 1, \dots, 1) = 1$$

i.e., using the notations (2.1), it follows from (2.6) that

$$[f(xy) + (k-1)f(y)]f(\mu(x)) = f(x)f(\mu(x)y).$$

Dividing both sides by  $f(x)f(\mu(x))f(y) \neq 0$  we obtain the desired equation (2.2).  $\square$

**3. The elimination of  $\mu$ .** In the present section we deduce a functional equation that contains only the unknown function  $f(x) = E(x, 1)$ . We shall need the following definition (DARÓCZY [3]).

*Definition.* The function  $E: \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is called a differentiable deviation on  $\mathbb{R}_+$  if it satisfies the following properties:

(E\*1) For each fixed  $x, y \in \mathbf{R}_+$

$$E_2(x, y) := \frac{\partial E(x, y)}{\partial y}$$

exists and is negative.

(E\*2)  $E(x, x) = 0$  for any  $x \in \mathbf{R}_+$ .

Denote by  $\varepsilon^*(\mathbf{R}_+)$  the set of all differentiable deviations on  $\mathbf{R}_+$ . Then it is obvious that  $\varepsilon^*(\mathbf{R}_+) \subset \varepsilon(\mathbf{R}_+)$ .

**Lemma.** Let  $E \in \varepsilon^*(\mathbf{R}_+)$  and suppose  $\mathfrak{M}_{n,E}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) to be a homogeneous mean. Then, using the notation  $f(x) := E(x, 1)$ , ( $x \in \mathbf{R}_+$ ), the function

$$(3.1) \quad x \rightarrow \frac{f(xy)}{f(x)f(y)} - \frac{f(xz)}{f(x)f(z)} =: G(x, y, z)$$

is continuous on the set  $\mathbf{R}_+ \setminus \{1\}$  and the limit exists at  $x=1$  for each fixed  $y, z \in \mathbf{R}_+ \setminus \{1\}$ .

**PROOF.** Since  $\mathfrak{M}_{n,E}$  is a homogeneous mean, applying the second theorem mentioned in the introduction, we obtain that there exist  $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  with  $h(1)=1$  such that

$$(3.2) \quad E(x, y) = h(y) f\left(\frac{x}{y}\right)$$

for  $x, y \in \mathbf{R}_+$ . With the help of this relation we get

$$\begin{aligned} G(x, y, z) &= \frac{E\left(y, \frac{1}{x}\right)}{E\left(1, \frac{1}{x}\right) E(y, 1)} - \frac{E\left(z, \frac{1}{x}\right)}{E\left(1, \frac{1}{x}\right) E(z, 1)} = \\ &= \frac{1}{E(y, 1)} \frac{E\left(y, \frac{1}{x}\right) - E(y, 1)}{E\left(1, \frac{1}{x}\right) - E(1, 1)} - \frac{1}{E(z, 1)} \frac{E\left(z, \frac{1}{x}\right) - E(z, 1)}{E\left(1, \frac{1}{x}\right) - E(1, 1)} \end{aligned}$$

for  $x, y, z \in \mathbf{R}_+ \setminus \{1\}$ . It follows from the last expression that  $x \rightarrow G(x, y, z)$  is a continuous function, and  $\lim_{x \rightarrow 1} G(x, y, z)$  exists since  $E \in \varepsilon^*(\mathbf{R}_+)$ .  $\square$

**Theorem 2.** Let  $E \in \varepsilon^*(\mathbf{R}_+)$  and assume that  $\mathfrak{M}_{n,E}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) is a  $k$ -homogeneous mean for some fixed  $k \geq 2$ . Then, using the notation  $f(x) = E(x, 1)$ ,

$$(3.3) \quad 2 \frac{f(xy)}{f(x)f(y)} = \frac{f(x^2)}{f^2(x)} + \frac{f(y^2)}{f^2(y)}$$

for any  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .

PROOF. Let  $y, z \in \mathbf{R}_+ \setminus \{1\}$  be fixed values and define the function  $\tilde{G}_{y,z}: \mathbf{R}_+ \rightarrow \mathbf{R}$  with the help of  $G$  as follows

$$\begin{aligned}\tilde{G}_{y,z}(x) &:= G(x, y, z), \quad x \in \mathbf{R}_+ \setminus \{1\}, \\ &:= \lim_{t \rightarrow 1} G(t, y, z), \quad x = 1.\end{aligned}$$

Then, by the Lemma,  $\tilde{G}_{y,z}$  is a continuous function. Now we prove that  $\tilde{G}_{y,z}$  is identically constant i.e.  $\tilde{G}_{y,z}(x) = \tilde{G}_{y,z}(1)$  for any  $x \in \mathbf{R}_+$ .

Let  $x_0 > 1$  be an arbitrary but fixed value, further let

$$H_{x_0} := \{x \in [1, x_0] \mid \tilde{G}_{y,z}(x) = \tilde{G}_{y,z}(x_0)\}.$$

Then the continuity of  $\tilde{G}_{y,z}$  implies that  $H_{x_0}$  is closed, and since  $x_0 \in H_{x_0}$ ,  $H_{x_0}$  is nonvoid.

Now apply the functional equation (2.2) for the values  $x, y$  and  $x, z$ . Subtracting the equations obtained we get

$$\tilde{G}_{y,z}(x) = \tilde{G}_{y,z}(\mu(x))$$

for  $x \in \mathbf{R}_+$ . It follows from this relation that

$$(3.4) \quad \mu(H_{x_0}) \subset H_{x_0}.$$

Denote by  $\bar{x}_0$  the greatest lower bound of the set  $H_{x_0}$ . Since  $H_{x_0}$  is closed we have  $\bar{x}_0 \in H_{x_0}$ . Therefore, by (3.4), we obtain

$$(3.5) \quad \mu(\bar{x}_0) \in H_{x_0}.$$

Now we prove that  $1 < \bar{x}_0$  cannot be valid. By the definition of  $\mu(\bar{x}_0)$  we have

$$E(\bar{x}_0, \mu(\bar{x}_0)) + (k-1)E(1, \mu(\bar{x}_0)) = 0.$$

Since  $k \geq 2$ , it follows from this relation that

$$(3.6) \quad 1 < \mu(\bar{x}_0) < \bar{x}_0$$

provided that  $1 < \bar{x}_0$ . However (3.6) and (3.5) contradict the definition of  $\bar{x}_0$ . Thus we have proved  $1 = \bar{x}_0$  i.e.  $1 \in H_{x_0}$ . Therefore  $\tilde{G}_{y,z}(x_0) = \tilde{G}_{y,z}(1)$ .

If  $x_0 < 1$  then it can analogously be seen that  $\tilde{G}_{y,z}(x_0) = \tilde{G}_{y,z}(1)$  is also satisfied.

Now notice the relation  $\tilde{G}_{y,z}(1) = -\tilde{G}_{z,y}(1)$  if  $y, z \in \mathbf{R}_+ \setminus \{1\}$ . By its help we obtain the equation

$$(3.7) \quad G(x, y, x) = \tilde{G}_{y,x}(x) = \tilde{G}_{y,x}(1) = -\tilde{G}_{x,y}(1) = -\tilde{G}_{x,y}(y) = -G(y, x, y),$$

for  $x, y \in \mathbf{R}_+ \setminus \{1\}$ . Taking into consideration the notion of  $G$  we get at once (3.3) from (3.7).  $\square$

**4. The solution of (3.3).** In this section we determine all the solutions  $f: \mathbf{R}_+ \rightarrow \mathbf{R}$  of (3.3) having the property  $\operatorname{sgn} f(x) = \operatorname{sgn}(x-1)$ ,  $x \in \mathbf{R}_+$ .

**Theorem 3.** Assume that the function  $f: \mathbf{R}_+ \rightarrow \mathbf{R}$  satisfies  $\operatorname{sgn} f(x) = \operatorname{sgn}(x-1)$  for  $x \in \mathbf{R}_+$  and (3.3) for  $x, y \in \mathbf{R}_+ \setminus \{1\}$ . Then there exists a constant  $b \geq 0$  such

that, for the function

$$(4.1) \quad g(x) = \frac{f(x^2)}{f^2(x)}, \quad x \in \mathbf{R}_+ \setminus \{1\},$$

we have

$$(4.2) \quad g(xy)[g(x) + g(y)] - g(x)g(y) = b^2$$

for any  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .

PROOF. Let, for  $x, y \in \mathbf{R}_+ \setminus \{1\}$ ,

$$F(x, y) := 2 \frac{f(xy)}{f(x)f(y)}.$$

Then it is obvious that  $F$  satisfies the equation

$$(4.3) \quad F(xy, z)F(x, y) = F(x, yz)F(y, z)$$

On the other hand, by (3.3), we have

$$F(x, y) = g(x) + g(y).$$

Therefore it follows from (4.3) that

$$[g(xy) + g(z)][g(x) + g(y)] = [g(x) + g(yz)][g(y) + g(z)]$$

for  $x, y, z \in \mathbf{R}_+ \setminus \{1\}$ . Using the notation

$$G(x, y) := g(xy)[g(x) + g(y)] - g(x)g(y)$$

we obtain

$$(4.4) \quad G(x, y) = G(y, z)$$

for  $x, y, z \in \mathbf{R}_+ \setminus \{1\}$ . The repeated application of (4.4) yields

$$(4.5) \quad G(x, y) = G(y, z) = G(z, u)$$

for  $x, y, z, u \in \mathbf{R}_+ \setminus \{1\}$ . It is obvious from (4.5) that  $G$  is identically constant, i.e. there exists  $c \in \mathbf{R}$  so that

$$(4.6) \quad g(xy)[g(x) + g(y)] - g(x)g(y) = c$$

if  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .

To complete the proof of the theorem it is enough to show that  $c \geq 0$ . Then  $c = b^2$  for a suitable  $b \geq 0$ .

Assume, on the contrary, that  $c < 0$ . Then, by (4.6),

$$g(xy)g(x) < g(y)[g(x) - g(xy)].$$

If  $x, y > 1$  then  $g(x), g(y), g(xy) > 0$  therefore we get

$$(4.7) \quad g(x) > g(xy).$$

Substituting into (4.7)  $x := x^n, y := x$  we obtain

$$g(x^n) > g(x^n x) = g(x^{n+1}).$$

On the other hand  $g(x^n) > 0$  hence the following limit exists

$$\lim_{n \rightarrow \infty} g(x^n) =: \bar{g}(x) \cong 0.$$

Substituting  $y := x^n$  into (4.6) and calculating the limit as  $n \rightarrow \infty$ , we get

$$\bar{g}^2(x) = c.$$

However this is a contradiction, since  $c < 0$ .  $\square$

First we discuss the functional equation (4.2) in the case  $b = 0$ .

**Theorem 4.** Assume that the function  $g: ]1, \infty[ \rightarrow \mathbf{R}_+$  satisfies the equation

$$(4.8) \quad g(xy)[g(x) + g(y)] - g(x)g(y) = 0$$

for  $x, y > 1$ . Then there exists a positive constant  $d$  such that

$$(4.9) \quad g(x) = \frac{1}{d \ln x}$$

for  $x > 1$ .

PROOF. Using the notation

$$l(x) := \frac{1}{g(x)}, \quad x > 1,$$

we obtain from (4.8) that

$$l(xy) = \frac{1}{g(xy)} = \frac{g(x) + g(y)}{g(x)g(y)} = \frac{1}{g(x)} + \frac{1}{g(y)} = l(x) + l(y)$$

if  $x, y > 1$ . Let, for  $t > 0$ ,

$$A(t) = l(e^t).$$

Then we have

$$A(t+s) = A(t) + A(s)$$

for  $t, s > 0$ . It is well known that there exists an additive function  $\bar{A}: \mathbf{R} \rightarrow \mathbf{R}$  such that  $\bar{A}(t) = A(t)$  if  $t > 0$  (see ACZÉL—ERDŐS [2], DARÓCZY—LOSONCZI [5]). On the other hand  $\bar{A}(t) = A(t) > 0$  if  $t \in \mathbf{R}_+$ , therefore there exists a constant  $d > 0$  such that  $\bar{A}(t) = dt$  for  $t \in \mathbf{R}$  (see ACZÉL [1]). However, for  $t > 0$ ,  $l(e^t) = dt$ , i.e. substituting  $l = \ln x$ , we obtain

$$l(x) = d \ln x$$

for  $x > 1$ . Therefore we just get the solution (4.9).  $\square$

**Theorem 5.** Let  $b > 0$ , and assume that  $g: ]1, \infty[ \rightarrow \mathbf{R}_+$  satisfies the equation

$$(4.10) \quad g(xy)[g(x) + g(y)] - g(x)g(y) = b^2$$

for  $x, y > 1$ . Then either

$$(i) \quad g(x) = b, \quad x > 1,$$

or

(ii) *there exists a positive constant  $a$  such that*

$$g(x) = b \frac{x^a + 1}{x^a - 1}$$

for  $x > 1$ .

PROOF. For  $x > 1$ , let

$$(4.11) \quad m(x) = \frac{g(x) - b}{g(x) + b}.$$

Then, by (4.10), we obtain

$$m(xy) = \frac{g(xy) - b}{g(xy) + b} = \frac{\frac{b^2 + g(x)g(y)}{g(x) + g(y)} - b}{\frac{b^2 + g(x)g(y)}{g(x) + g(y)} + b} = \left[ \frac{g(x) - b}{g(x) + b} \right] \left[ \frac{g(y) - b}{g(y) + b} \right] = m(x)m(y)$$

for  $x, y > 1$ .

The function  $g(x) \equiv b$  is obviously a solution of (4.10). Now try to find all the different solutions. Assume that  $g(y_0) \neq b$  for some  $y_0 > 1$ . Then we prove that  $g(x) \neq b$  for  $x > 1$ . Choose  $n \in \mathbb{N}$  such that  $y_0^n > x$ . Then, since  $g(y_0) \neq b$ ,  $m(y_0) \neq 0$ , we have  $0 \neq m^n(y_0) = m(y_0^n) = m\left(\frac{y_0^n}{x}x\right) = m\left(\frac{y_0^n}{x}\right)m(x)$ . Hence  $m$  does not vanish anywhere. Then  $m$  is positive, since

$$m(x) = m(\sqrt{x}\sqrt{x}) = m^2(\sqrt{x}) > 0.$$

Rearranging (4.11), we obtain

$$(4.12) \quad g(x) = b \frac{1 + m(x)}{1 - m(x)}$$

for  $x > 1$ . Since  $g$  is positive, we have  $m(x) < 1$  for  $x > 1$ . Let

$$A(t) = \ln m(e^t)$$

if  $t > 0$ . Then it follows from the properties of  $m$  that  $A(t+s) = A(t) + A(s)$  for  $t, s > 0$  and  $A(t) < 0$  if  $t < 0$ . Therefore, as we have shown in the proof of Theorem 4, there exists a positive constant  $a$  such that  $A(t) = -at$  for  $t > 0$ . Substituting  $t = \ln x$  we have

$$\ln m(x) = -a \ln x$$

i.e.

$$m(x) = x^{-a}$$

for  $x > 1$ . Applying (4.12) we obtain immediately the solution (ii).  $\square$

**Theorem 6.** *Let  $f: \mathbf{R}_+ \rightarrow \mathbf{R}$  and assume that  $\operatorname{sgn} f(x) = \operatorname{sgn}(x-1)$  if  $x \in \mathbf{R}_+$ , further  $f$  satisfies the equation (3.3) for  $x, y \in \mathbf{R}_+ \setminus \{1\}$ . Then there exist a multiplicative function  $m: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  (i.e.  $m(xy) = m(x)m(y)$  if  $x, y \in \mathbf{R}_+$  and positive constants  $a, c$  such that either*

$$(j) \quad f(x) = cm(x) \ln x$$

or

$$(jj) \quad f(x) = cm(x)(x^a - 1) \\ \text{for } x \in \mathbf{R}_+.$$

PROOF. Using the notation (4.1) we easily obtain from (3.3) that

$$(4.13) \quad 2 \frac{f(xy)}{f(x)f(y)} = g(x) + g(y)$$

if  $x, y \in \mathbf{R}_+ \setminus \{1\}$ . Further, by Theorem 3,  $g$  satisfies the equation (4.2) for  $x, y \in \mathbf{R}_+ \setminus \{1\}$  with a suitable constant  $b \equiv 0$ .

If  $b=0$ , then, by Theorem 4, there exists a constant  $d>0$  such that  $g(x) = \frac{1}{d \ln x}$ , ( $x>1$ ). We prove this equation also for  $0<x<1$ . Let  $0<x<1$  and  $y=1/x$  in the equation (4.13). Since  $f(xy)=f(x(1/x))=f(1)=0$ , it follows from (4.13) that

$$g(x) = -g\left(\frac{1}{x}\right) = -\frac{1}{d \ln\left(\frac{1}{x}\right)} = \frac{1}{d \ln x}.$$

Therefore

$$(4.14) \quad 2 \frac{f(xy)}{f(x)f(y)} = \frac{1}{d \ln x} + \frac{1}{d \ln y} = \frac{d \ln xy}{(d \ln x)(d \ln y)}$$

for any  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .

With the help of (4.14) it is easy to see that the function  $m: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  defined by

$$m(x) := \frac{2f(x)}{d \ln x}, \quad x \in \mathbf{R}_+ \setminus \{1\}, \\ := 1, \quad x = 1$$

is multiplicative. Hence, for  $x \in \mathbf{R}_+$ ,

$$f(x) = \frac{d}{2} m(x) \ln x = cm(x) \ln x.$$

If  $b>0$ , then by Theorem 5 we have two possibilities. If  $g(x) \equiv b$  for any  $x>1$ , then applying (4.13) for  $0<x<1$ ,  $y=1/x$  we get

$$g(x) = -g\left(\frac{1}{x}\right) = -b.$$

But then, by (4.13),

$$2 \frac{f\left(4 \cdot \frac{1}{2}\right)}{f(4)f\left(\frac{1}{2}\right)} = g(4) + g\left(\frac{1}{2}\right) = b - b = 0,$$

hence  $f(2)=0$ . Because of this contradiction there remains the case

$$g(x) = b \frac{x^a + 1}{x^a - 1}, \quad x > 1.$$

Then, it follows from (4.13) that

$$g(x) = -g\left(\frac{1}{x}\right) = -b \frac{x^{-a} + 1}{x^{-a} - 1} = b \frac{x^a + 1}{x^a - 1}$$

for  $0 < x < 1$ . Applying (4.13) again we get

$$(4.15) \quad \frac{2f(xy)}{f(x)f(y)} = b \frac{x^a + 1}{x^a - 1} + b \frac{y^a + 1}{y^a - 1} = \frac{2b[(xy)^a - 1]}{(x^a - 1)(y^a - 1)}$$

if  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .

It follows from (4.15) that the function  $m: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  defined by

$$m(x) := \frac{bf(x)}{x^a - 1}, \quad x \in \mathbf{R}_+ \setminus \{1\},$$

$$:= 1, \quad x = 1$$

is multiplicative. Hence

$$f(x) = \frac{1}{b} m(x)(x^a - 1) = cm(x)(x^a - 1)$$

for  $x \in \mathbf{R}_+$ .

Conversely, it can easily be checked that the obtained functions (j) and (jj) really satisfy the functional equation (3.3) for  $x, y \in \mathbf{R}_+ \setminus \{1\}$ .  $\square$

## 5. $k$ -homogeneous deviation means

**Theorem 7.** Let  $E \in \varepsilon^*(\mathbf{R}_+)$  and assume that  $\mathfrak{M}_{n,E}: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) is a  $k$ -homogeneous mean for some fixed  $k \geq 2$ . Then there exist a multiplicative function  $m: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  (i.e.  $m(xy) = m(x)m(y)$  for  $x, y \in \mathbf{R}_+$ ) and a positive constant  $a$  such that either (1.11) or (1.12) is satisfied.

Conversely, the means obtained are really  $k$ -homogeneous, moreover they are multiplicative means.

**PROOF.** If  $E \in \varepsilon^*(\mathbf{R}_+)$  and  $\mathfrak{M}_{n,E}$  is  $k$ -homogeneous then, by Theorem 2,  $f(x) = E(x, 1)$  satisfies equation (3.3) for  $x, y \in \mathbf{R}_+ \setminus \{1\}$ . Hence, by Theorem 6, there exist a multiplicative function  $m$  and positive constants  $a, c$  such that either (j) or (jj) is satisfied. On the other hand  $\mathfrak{M}_{n,E}$  is a homogeneous mean, therefore there exists a function  $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  such that  $h(1) = 1$  and (3.2) is valid. Hence we obtain

that either

$$\begin{aligned} E(x, y) &= h(y)f\left(\frac{x}{y}\right) = h(y)cm\left(\frac{x}{y}\right)\ln\frac{x}{y} = \\ &= c\frac{h(y)}{m(y)}m(x)(\ln x - \ln y) = \\ &= H(y)m(x)(\ln(x) - \ln y) \end{aligned}$$

or

$$\begin{aligned} E(x, y) &= h(y)f\left(\frac{x}{y}\right) = h(y)cm\left(\frac{x}{y}\right)\left[\left(\frac{x}{y}\right)^a - 1\right] = \\ &= c\frac{h(y)}{m(y)y^a}m(x)(x^a - y^a) = H(y)m(x)(x^a - y^a). \end{aligned}$$

Now let  $x_1, \dots, x_n \in \mathbf{R}_+$  ( $n \in \mathbf{N}$ ) and consider the equation

$$\sum_{i=1}^n E(x_i, y) = 0.$$

Solving this equation for  $y$  in both cases we easily get that either (1.11) or (1.12) is valid.

The multiplicativity of  $\mathfrak{M}_{n,E}$  of this form can easily be checked.  $\square$

*Remark.* Apparently, in the above manner, we have obtained the means in (1.12) only for  $a > 0$ . However, for  $a < 0$

$$\left[ \frac{\sum_{i=1}^n m(x_i)x_i^a}{\sum_{i=1}^n m(x_i)} \right]^{1/a} = \left[ \frac{\sum_{i=1}^n \bar{m}(x_i)x_i^{-a}}{\sum_{i=1}^n \bar{m}(x_i)} \right]^{-1/a}$$

where  $\bar{m}(x) = m(x)x^a$  is also a multiplicative function.

**6. Open problems.** In the present paper we have proved that the  $k$ -homogeneity of a deviation mean and certain regularity assumptions imply the multiplicativity of the mean. It would have some interest to find a value  $k$  and a deviation such that the generated mean is  $k$ -homogeneous but not multiplicative. It can be proved that the  $k$ -homogeneity of deviation means implies  $l$ -homogeneity if  $l \leq k$ . Therefore if there exists a nonmultiplicative but  $k$ -homogeneous (for some  $k$ ) deviation mean, then necessarily there also exists a nonmultiplicative 2-homogeneous mean.

In our discussion the functional equation (2.2) plays an important role. It would be very useful to know all the solutions  $f: \mathbf{R}_+ \rightarrow \mathbf{R}$  with  $\operatorname{sgn} f(x) = \operatorname{sgn}(x-1)$ , ( $x \in \mathbf{R}_+$ ), and  $\mu: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  with  $\operatorname{sgn}(\mu(x)-1) = \operatorname{sgn}(x-1)$  ( $x \in \mathbf{R}_+$ ) of equation (2.2).

The regularity properties of  $E$  were used only in the elimination of  $\mu$ . It is easy to see that we would get all the  $k$ -homogeneous deviation means if we could deduce equation (3.1) without using regularity properties for the deviations.

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