

On Lie algebras connected with associative *PI*-algebras

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1. Let $\mathfrak{X}$ be a nontrivial variety of associative algebras over a field k of characteristic zero. Following [1] denote by $\tilde{\mathfrak{X}}$ the class of Lie algebras which possess enveloping algebras belonging to $\mathfrak{X}$ (such Lie algebras were called special or *SPI*-algebras in [2]). It is easy to see that $\tilde{\mathfrak{X}}$ is a quasi-variety of Lie algebras (cf. [1]). A question was posed in [2] whether a homomorphic image of an *SPI*-algebra is an *SPI*-algebra itself. In other words: does the variety of Lie algebras generated by $\tilde{\mathfrak{X}}$ belong to $\tilde{\mathfrak{Y}}$ for some nontrivial variety of associative algebras $\mathfrak{Y}$. In this note we prove that:

- (1) a quasi-variety $\tilde{\mathfrak{X}}$ need not be a variety;
- (2) all homogeneous algebras of a variety generated by an *SPI*-algebra are *SPI*-algebras (see also [8]).

2. We recall some notations and definitions from [3].

Let $U(\varrho)$ be a universal enveloping algebra of a Lie algebra ϱ . A homomorphism $\varphi: \varrho \rightarrow \mathfrak{h}$ induces a homomorphism $\varphi_*: U(\varrho) \rightarrow U(\mathfrak{h})$. The ideal $\text{Ker } \varphi_*$ is generated by $\text{Ker } \varphi$ and will be denoted by $\omega \text{Ker } \varphi$. Let $\eta: S(\varrho) \rightarrow U(\varrho)$ be the linear isomorphism of the Poincaré—Birkhoff—Witt theorem. For $a_1, a_2, \dots, a_n \in \varrho$

let $\eta(a_1 a_2 \dots a_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)}$, where $\mathfrak{S}_n$ is the symmetric group of the

set $\{1, 2, \dots, n\}$. $U(\varrho) = \bigoplus_{i=0}^{\infty} U^i(\varrho)$, where $U^i(\varrho)$ is an image of the subspace $S^i(\varrho) \subset S(\varrho)$ of homogeneous polynomials of degree i . If $\varphi: \varrho \rightarrow \mathfrak{h}$ is a homomorphism of Lie algebras then $\varphi_*(U^n(\varrho)) \subset U^n(\mathfrak{h})$.

A free associative algebra $A(X)$ with a (countable) set of free generators $X = \{x_1, x_2, \dots\}$ is a universal enveloping algebra of its Lie subalgebra $L(X)$ generated by X (the commutator brackets are used as usual: $[a, b] = ab - ba$). $L(X)$ is a free Lie algebra freely generated by X . $A(X)$ is naturally graded by the degrees of its elements which are (noncommutative) polynomials in free variables. An ideal $H \subset L(X)$ (or $A(X)$) is called homogeneous if it is generated by homogeneous elements with respect to this grading. A Lie algebra $L(X)/H$ is called homogeneous if H is a homogeneous ideal in $L(X)$. An ideal $\omega H \subset A(X)$ is (obviously) homogeneous if H is homogeneous.

Let $A^+(X) \subset A(X)$ be a set of polynomials with zero constant terms. A linear mapping $\pi: A^+(X) \rightarrow L(X)$ is defined on monomials by the formula $(x_{i_1} x_{i_2} \dots x_{i_n}) \pi =$

$=(\text{ad}(x_{i_1}) \circ \text{ad}(x_{i_2}) \circ \dots \circ \text{ad}(x_{i_n})) \cdot (x_{i_{n-1}})$. If $v \in L(X)$ is homogeneous of degree m then $v\pi = mv$. The following lemma is obvious (it implies that π can be defined on universal enveloping algebras of homogeneous Lie algebras).

Lemma 1. *If H is a homogeneous ideal in $L(X)$ then $(\omega H)\pi = H$.*

3. We need also some basic facts about verbal ideals in universal enveloping algebras (cf. [4]). An ideal $I \subset A(X)$ is said to be verbal if it is invariant relative to all the endomorphisms of $A(X)$ induced by the endomorphisms of $L(X)$. One can easily verify that verbal ideals in $A(X)$ are homogeneous (linearization). It is also clear from the definition that T -ideals are verbal. Note also, that an intersection of a verbal ideal of $A(X)$ with $L(X)$ is a verbal ideal of $L(X)$ in the usual sense. Now, let $U(\varrho)$ be a universal enveloping algebra of a Lie algebra ϱ . An ideal in $U(\varrho)$ is said to be verbal if it consists of the elements of the form v^μ , where v runs over a certain verbal ideal $I \subset A(X)$ and μ runs over all the homomorphisms $A(X) \rightarrow U(\varrho)$ induced by the homomorphisms $L(X) \rightarrow \varrho$. Denote by I_ϱ a verbal ideal in $U(\varrho)$ which corresponds to a verbal ideal $I \subset A(X)$. It is easy to see that if $\psi: \varrho \rightarrow \mathfrak{h}$ is an epimorphism then $\psi_*(I_\varrho) = I_\mathfrak{h}$. Finally, suppose that I is a T -ideal of identities which are satisfied by a variety of associative algebras $\mathfrak{A}$. Another obvious fact is

Lemma 2. $\varrho \in \tilde{\mathfrak{A}}$ if and only if $\varrho \cap I_\varrho = \{0\}$.

4. **Proposition 1.** *There exist varieties of associative algebras $\mathfrak{A}$ for which quasivarieties $\tilde{\mathfrak{A}}$ are not varieties of Lie algebras.*

PROOF. Let $\varrho \in \tilde{\mathfrak{A}}$. By lemma 2 $I_\varrho \cap \varrho = \{0\}$. Let $\varphi: \varrho \rightarrow \mathfrak{h}$ be an epimorphism. Suppose that we can choose $u = v + w \in I_\varrho$ with $v \in \varrho$ and $w \in \bigoplus_{i \geq 2} U^i(\varrho)$ such that $\varphi(v) \neq 0$ and $\varphi_*(w) = 0$. Then $0 \neq \varphi(v) = \varphi_*(u) \in I_\mathfrak{h} \cap \mathfrak{h}$ and $\mathfrak{h} \notin \tilde{\mathfrak{A}}$. We see that it is sufficient to find a variety of associative algebras $\mathfrak{A}$, a Lie algebra $\varrho \in \tilde{\mathfrak{A}}$, an ideal $\varrho_1 \subset \varrho$ and an element $v_1 + v_2 \in I_\varrho$ such that $v_1 \in \varrho$, $v_2 \in \bigoplus_{i \geq 2} U^i(\varrho)$, $v_2 \in \omega \varrho_1$ and $v_1 \notin \varrho_1$. Let us begin with the free algebra $A(X) = U(L(X))$. The set P_n of multilinear polynomials in n variables $x_1, x_2, \dots, x_n$ is a left and right $k\mathfrak{S}_n$ -module. Let $P_n^i = U^i(L(X)) \cap P_n$, $i = 1, 2, \dots, n$. Clearly, the P_n^i are left (but not right) $k\mathfrak{S}_n$ -modules. The mapping π restricted to P_n acts as a right multiplication by an element $\pi_n \in k\mathfrak{S}_n$ [5]. Let $I \subset A(X)$ be the T -ideal generated by the element $u(1 + \alpha\pi_n)$, where $u \in P_n^2$ and $\alpha \in k\mathfrak{S}_n$. Let also $L_0 = L(X) \cap I$.

Lemma 3. $\omega L_0 \cap P_n = \{0\}$.

PROOF. First, observe that $\omega L_0 \cap P_n = L_0 \cap P_n$, as, clearly, $L_0 \cap P_i = \{0\}$ for $i < n$. Now, $I \cap P_n = k\mathfrak{S}_n u(1 + \alpha\pi_n)$ and if $\beta u + \beta u \alpha \pi_n \in L(X)$ for some $\beta \in k\mathfrak{S}_n$ then $\beta u = 0$ and $\beta u \alpha \pi_n = 0$.

Lemma 2 and the remarks that precede it show that $L_1 = L(X)/L_0 \in \tilde{\mathfrak{A}}$. In view of lemma 3, there remains to find a homogeneous ideal $H \in L(X)$ such that $u\alpha\pi_n \notin H$ and $u \in \omega H$. An example can be found for $n = 4$. Let $u = [[x_1, x_2], x_3]x_4 + x_4[[x_1, x_2], x_3]$. Let the ideal $H \subset L(X)$ be generated by $[[x_1, x_2], x_3]$ and let

$\alpha \in \mathfrak{S}_4$ be the cycle (4321). A straightforward calculation shows that

$$\begin{aligned} u\alpha\pi_4 \equiv & -2 [[x_1, x_2], [x_3, x_4]] + [[x_1, x_3], [x_2, x]] + \\ & + [[x_1, x_4], [x_2, x_3]] - 2 [x_3, [x_2, [x_1, x_4]]] \pmod{H}. \end{aligned}$$

Using Hall's basis in $L(X)$ (see [3]) one can easily verify that $u\alpha\pi_4 \notin H$. This proves proposition 1.

5. The following lemma may be interesting on its own.

Lemma 4. *For any associative PI-algebra A with a unity there exists a nontrivial variety of associative algebras $\mathfrak{V}$ such that if $u \in A(X)$ is an identity of $\mathfrak{V}$ then $u\pi$ is an identity of A .*

PROOF. Let A^0 be the algebra opposite to A and let $B = A^0 \otimes A$. The vector space $\tilde{A}$ of A can be endowed with a B - B -bimodule structure if we define $(a \otimes b) \cdot c = acb$ and $c \cdot (a \otimes b) = 0$ for all $a, b, c \in A$. Convert $\tilde{A}$ into the algebra with trivial multiplication and take the semidirect sum $C = \tilde{A} \oplus B$ (the multiplication in C is given by the rule $(a_1 + b_1)(a_2 + b_2) = b_1 a_2 + b_1 b_2$, where $a_1, a_2 \in \tilde{A}$; $b_1, b_2 \in B$). A theorem of A. REGEV [6] shows that C is a PI-algebra. Let $c_i = a_i + b_i$, $a_i \in \tilde{A}$, $b_i \in B$; $i = 1, 2, \dots, n$. One easily obtains that $c_1 c_2 \dots c_n = b_1 b_2 \dots b_{n-1} a_n + b_1 b_2 \dots b_n$. It is clear now that if $u = u(x_1, x_2, \dots, x_n) = \sum_{i_1, i_2, \dots, i_m} \lambda_{i_1 i_2 \dots i_m} X_{i_1} X_{i_2} \dots X_{i_m} \in A(X)$ then $u(c_1, c_2, \dots, c_n) = \sum_{i_1, i_2, \dots, i_m} \lambda_{i_1 i_2 \dots i_m} b_{i_1} b_{i_2} \dots b_{i_{m-1}} a_{i_m} + u(b_1, b_2, \dots, b_n)$. If u is an identity of C then $u(b_1, b_2, \dots, b_n) = 0$ and $u(c_1, c_2, \dots, c_n) = \sum_{i_1, i_2, \dots, i_m} \lambda_{i_1 i_2 \dots i_m} b_{i_1} b_{i_2} \dots b_{i_{m-1}} a_{i_m}$ for all $c_i \in C$. Set $b_j = 1 \otimes a_j - a_j \otimes 1$, $j = 1, 2, \dots, n$. Obviously, $0 = u(c_1, c_2, \dots, c_n) = \sum_{i_1, i_2, \dots, i_m} \lambda_{i_1 i_2 \dots i_m} (\text{ad}(a_{i_1}) \circ \dots \circ \text{ad}(a_{i_{m-1}}))(a_{i_m}) = (u\pi)(a_1, a_2, \dots, a_n)$, where the right hand side is calculated in A .

Proposition 2. *Any homogeneous Lie algebra from a variety generated by a special Lie algebra is itself special.*

PROOF. Let A be an associative enveloping PI-algebra of a special Lie algebra L . One can assume that A possesses a unit element. We will show that a homogeneous algebra L_1 which belongs to the variety generated by L is contained also in $\mathfrak{V}$, where $\mathfrak{V}$ is the variety obtained from A with the use of the construction of Lemma 4. Let $L_1 = L(X)/H$, where H is a homogeneous ideal in $L(X)$. Note, that $H \supset \{\text{identities of } L\}$. Suppose that $L_1 \notin \mathfrak{V}$. This means that some homogeneous element $f \in L(X) \setminus H$ equals $u + a$ in $A(X)$, where u is an identity of $\mathfrak{V}$ and $a \in \omega H$. One obtains from this equality that $nf = u\pi + a\pi$, where $n = \deg f \neq 0$. But $u\pi \in H$ by lemma 4 and $a\pi \in H$ by lemma 1, QED.

Corollary. *A relatively free Lie algebra which belongs to a variety generated by a special Lie algebra is itself special.*

Concluding remarks. Proposition 1 suggests an interesting problem of characterizing varieties of associative algebras $\mathfrak{K}$ for which $\mathfrak{K}$ is a variety. On the other

hand, it is unknown to the author whether proposition 2 can be extended to cover the case of nonhomogeneous algebras. Some examples of SPI -algebras can be found in [2] and [7].

References

- [1] B. I. PLOTKIN, Varieties and quasi-varieties connected with representations of groups, *Dokl. Akad. Nauk SSSR* **196** (1971), 527—530 = *Soviet Math. Dokl.* **12** (1971), 192—196.
- [2] V. N. LATYSHEV, On Lie algebras with identical relations, *Sibirsk. Mat. Ž.* **4** (1963), 821—829.
- [3] N. BOURBAKI, Groupes et Algèbres de Lie, Chap. I, II, *Hermann, Paris*, 1971, 1972.
- [4] L. A. SIMONIAN, On correspondence between varieties of linear representations of groups and Lie algebras, *Sbornik rabot po algebre, Riga*, 1980, 258—298 (*russian*).
- [5] W. MAGNUS, A. KARRASS, D. SOLITAR, Combinatorial Group Theory, *Wiley & Sons*, 1966.
- [6] A. REGEV, Existence of identities in $A \otimes B$, *Isr. J. Math.*, 1972, **11**, 131—152.
- [7] YU. A. BACHTURIN, On Lie subalgebras of associative PI -algebras, *J. of Algebra* **67** (1980), 257—271.
- [8] S. A. PIHTILKOV, On special Lie algebras, *Uspehi Mat. Nauk* **36**, 1981, 225—226.

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