

On some properties of the stable sequences of random elements

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1. Let $\mathfrak{X}_S$ be the set of all random elements (r.e.) defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in a separable, complete metric space (S, ϱ) , i.e.

$$\mathfrak{X}_S = \{X: \Omega \rightarrow S, X^{-1}(\mathcal{B}) \subset \mathcal{A}\},$$

where $\mathcal{B}$ stands for the σ -field, generated by the open sets of S . When $S = \mathbf{R}$, we set

$$\mathfrak{X}_R = \{X: \Omega \rightarrow \mathbf{R}, X^{-1}((-\infty, x]) \in \mathcal{A}, x \in \mathbf{R}\},$$

for the set of all random variables (r.v.) defined on $(\Omega, \mathcal{A}, P)$.

Definition 1. A sequence $\{X_n, n \geq 1\}$ of r.e. is called stable if for every $B \in \mathcal{A}$, $P(B) > 0$, there exists a probability measure μ_B such that

$$\lim_{n \rightarrow \infty} P([X_n \in A] | B) = \mu_B(A)$$

for every $A \in \mathcal{C}_{\mu_B} = \{A \in \mathcal{B}: \mu_B(\partial A) = 0\}$, where ∂A denotes the boundary of A and $P(D|B) = \frac{P(D \cap B)}{P(B)}$.

If $\mu_B(A) = \mu(A)$ for every $B \in \mathcal{A}$, $P(B) > 0$, then the sequence $\{X_n, n \geq 1\}$ of r.e. is called μ -mixing.

It is easy to observe that the sequence $\{X_n, n \geq 1\}$ of r.v. is stable iff

$$\lim_{n \rightarrow \infty} P([X_n < x] | B) = F_B(x) \quad \text{for every } B \in \mathcal{A}, P(B) > 0,$$

and $x \in \mathcal{C}_{F_B}$, where $F_B(x)$ denotes the distribution function.

Let us denote

$$Q_A(B) = \mu_B(A)P(B), \quad B \in \mathcal{A}, P(B) > 0, \quad A \in \mathcal{B},$$

and

$$Q_x(B) = F_B(x)P(B), \quad B \in \mathcal{A}, P(B) > 0, \quad x \in \mathbf{R}.$$

The measure Q_A and Q_x are absolutely continuous with respect to the measure P and by the Radon—Nikodym theorem there exist density functions α_A and α_x such that

$$Q_A(B) = \int_B \alpha_A dP \quad \text{and} \quad Q_x(B) = \int_B \alpha_x dP, \quad B \in \mathcal{A}.$$

If α_A is a constant a.e. for every $A \in \mathcal{B}$, then the sequence $\{X_n, n \geq 1\}$ of r.e. is mixing. It is easy to see that the sequence $\{X_n, n \geq 1\}$ of r.v. is mixing iff α_x is a constant a.e. for every $x \in \mathbf{R}$.

The aim of this paper is to give some properties of the stable sequences of random elements taking values in a metric space. The main theorem (Theorem 4) extends results of [5] concerning the stability and the density function of a sequence $\{g(X_n, Y), n \geq 1\}$, where $\{X_n, n \geq 1\}$ is a stable sequence of random variables, g is a real function, and Y is a random vector to the case when $X_n, n \geq 1, Y$ are random elements taking values in some metric space and g is a continuous mapping. Moreover, we generalize some results of [1], [10], and [11].

2. First we shall give a characterization of a probability space in terms of the equivalence of some kind of convergence of a sequence $\{X_n, n \geq 1\}$ of r.e.

In what follows we shall use the well known result [3]: $X_n \xrightarrow{P} X, n \rightarrow \infty$ (P — in probability) iff

$$(1) \quad \lim_{n \rightarrow \infty} P([X_n \in A] \triangle [X \in A]) = 0 \quad \text{for every } A \in \mathcal{C}_{P_X},$$

where $A \triangle B$ denotes the symmetric difference of A and B .

It is well known [7] that any sample space Ω can be represented as

$$(2) \quad \Omega = B \cup \bigcup_{k=1}^{\infty} B_k, \quad B_m \cap B_n = \emptyset, \quad m \neq n, \quad B \cap B_m = \emptyset, \quad m = 1, 2, \dots,$$

each B_k is either an atom or an empty set, and B has the property that, for any given $A \in \mathcal{A}$ such that $A \subset B$ and any $\varepsilon, 0 < \varepsilon < P(A)$, there exists $C \in \mathcal{A}, C \subset A$, such that $P(C) = \varepsilon$. Random elements are constant on atoms.

Theorem 1. *The following statements are equivalent:*

- a) every stable sequence $\{X_n, n \geq 1\}$ of r.e. converges in probability;
- b) for any r.e. X and any sequence $\{X_n, n \geq 1\}$ of r.e. $X_n \xrightarrow{P} X, n \rightarrow \infty$ iff $X_n \xrightarrow{\text{a.s.}} X, n \rightarrow \infty$;
- c) Ω is at most a countable union of disjoint atoms.

PROOF. The part b) $\Leftrightarrow$ c) has been proved in [10].

c) $\Rightarrow$ a). Let $\Omega = \bigcup_{i=1}^{\infty} B_i$ and $\{X_n, n \geq 1\}$ be a stable sequence of r.e. Then

$$P([X_n \in A] \cap B_i) \rightarrow \mu_{B_i}(A) P(B_i), \quad B_i \in \mathcal{A}, \quad A \in \mathcal{B}.$$

If

$$X_n(\omega) = a_{n,i} \quad \text{for } \omega \in B_i, \quad i = 1, 2, \dots, \quad n = 1, 2, \dots,$$

then

$$[X_n \in A] \cap B_i = \begin{cases} B_i & \text{if } a_{n,i} \in A, \\ 0 & \text{if } a_{n,i} \notin A. \end{cases}$$

For every $A \in \mathcal{C}_{\mu_{B_i}}$ the limit

$$\lim_{n \rightarrow \infty} P([X_n \in A] \cap B_i)$$

exists, so there exists n_0 such that $a_{n,i} \in A$ for all $n \geq n_0$ or $a_{n,i} \notin A$ for all $n \geq n_0$, which proves that the sequence $\{a_{n,i}, n \geq 1\}$ converges. Indeed, by the stability of the sequence $\{X_n, n \geq 1\}$ we know that $\{X_n, n \geq 1\}$ is tight, i.e. for any given $\varepsilon > 0$ there exists a compact set $K \subset S$ such that

$$\sup_n P[X_n \notin K] < \varepsilon.$$

Therefore, for every $i \in N$ we can choose a compact set $K_i \subset S$ such that $\sup_n P([X_n \notin K_i] \cap B_i) < \frac{1}{4} P(B_i)$ and $\mu_{B_i}(K_i) > \frac{3}{4}$. Thus $a_{n,i} \in K_i, n = 1, 2, \dots$. Assume that the sequence $\{a_{n,i}, n \geq 1\}$ does not converge as $n \rightarrow \infty$. Then there exist subsequences $\{a_{n_k,i}, k \geq 1\}$ and $\{a_{n'_k,i}, k \geq 1\}$ such that

$$a_{n_k,i} \rightarrow a_i, \quad k \rightarrow \infty, \quad a_{n'_k,i} \rightarrow a'_i, \quad k \rightarrow \infty, \quad \text{and} \quad \varrho(a_i, a'_i) > 0.$$

Putting now $A = \{s \in S: \varrho(s, a_i) < r\}$, where $r < \varrho(a_i, a'_i)$ is such that $A \in \mathcal{C}_{\mu_{B_i}}$ we see that $\lim_{n \rightarrow \infty} P([X_n \in A] \cap B_i)$ does not exist for every i which contradicts the assumption that $\{X_n, n \geq 1\}$ is a stable sequence. Let

$$a_i = \lim_{n \rightarrow \infty} a_{n,i}, \quad i = 1, 2, \dots,$$

and put

$$X(\omega) = a_i \quad \text{for} \quad \omega \in B_i.$$

Then

$$X_n \xrightarrow{\text{a.s.}} X, \quad n \rightarrow \infty.$$

a) $\Rightarrow$ c). Assume that $\Omega = B \cup \bigcup_{i=1}^{\infty} B_i$ and $P(B) > 0$. Then there exists a mixing sequence $\{D_n, n \geq 1\}$ of events such that $B \supset D_n \in \mathcal{A}$, $P(D_n) = \frac{1}{2} P(B)$, and

$$\lim_{n \rightarrow \infty} P(D_n \cap B') = \frac{1}{2} P(B'), \quad B' \subset B, \quad B' \in \mathcal{A}.$$

Indeed, let $B_0 \in \mathcal{A}$, $B_0 \subset B$ with $P(B_0) = \frac{1}{2} P(B)$, and put $B_1 = B \setminus B_0$. Now let $B_{i_1,0} \in \mathcal{A}$, $B_{i_1,0} \subset B_{i_1}$ with $P(B_{i_1,0}) = \frac{1}{2} P(B_{i_1})$, $i_1 = 0, 1$, and put $B_{i_1,1} = B_{i_1} \setminus B_{i_1,0}$. By the induction argument we see that there exist sets

$$B_{i_1, i_2, \dots, i_k, 0} \in \mathcal{A}, \quad B_{i_1, i_2, \dots, i_k, 0} \subset B_{i_1, i_2, \dots, i_k}$$

with

$$P(B_{i_1, i_2, \dots, i_k, 0}) = \frac{1}{2} P(B_{i_1, i_2, \dots, i_k}),$$

and

$$B_{i_1, i_2, \dots, i_k, 1} = B_{i_1, i_2, \dots, i_k} \setminus B_{i_1, i_2, \dots, i_k, 0}.$$

Setting

$$D_1 = B_1, \quad D_2 = B_{0,1} \cup B_{1,1}, \dots,$$

$$D_n = \bigcup_{\{(i_1, i_2, \dots, i_{n-1}): i_k=0 \text{ or } 1, k=1, 2, \dots, n-1\}} B_{i_1, i_2, \dots, i_{n-1}, 1}$$

we see that

$$P(D_n \cap D_k) = \frac{1}{2} P(D_k), \quad k < n,$$

which proves, by the Rényi theorem 3 of [8], the existence of the mixing sequence $\{D_n, n \geq 1\}$ given above. Let

$$X_n(\omega) = \begin{cases} a & \text{if } \omega \in D_n \\ b & \text{if } \omega \notin D_n, \quad a, b \in S, \quad \varrho(a, b) > 0. \end{cases}$$

It is easy to see that the sequence $\{X_n, n \geq 1\}$ of r.e. is stable but it does not converge in probability. Indeed, if

$$X(\omega) = \begin{cases} a, & \omega \in H \\ b, & \omega \notin H, \quad H \in \mathcal{A}, \quad H \subset B, \end{cases}$$

then

$$\lim_{n \rightarrow \infty} P \left[\varrho(X_n, X) \geq \frac{1}{2} \varrho(a, b) \right] = \lim_{n \rightarrow \infty} (P([X_n = b], [X = a]) +$$

$$+ P([X_n = a], [X = b])) = \frac{1}{2} P(B) > 0,$$

which proves that $X_n \xrightarrow{P} X, n \rightarrow \infty$. Thus Ω is at most a countable union of disjoint atoms.

3. In what follows we shall use the following

Lemma 1. Suppose that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are sequences of r.e. such that

$$(3) \quad \varrho(X_n, Y_n) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

If $\{X_n, n \geq 1\}$ is stable, then $\{Y_n, n \geq 1\}$ is also stable with the same density function.

PROOF. Note that for every B , $P(B) > 0$, and any given $A \in \mathcal{C}_{\mu_B}$, we have for any given $\varepsilon > 0$,

$$P([Y_n \in A] | B) = P([Y_n \in A] \cap [\varrho(X_n, Y_n) < \varepsilon] | B) + P([Y_n \in A] \cap [\varrho(X_n, Y_n) \geq \varepsilon] | B).$$

We see that

$$P([X_n \in \overline{A}^\varepsilon] \cap [\varrho(X_n, Y_n) < \varepsilon] | B) + P([Y_n \in A] \cap [\varrho(X_n, Y_n) \geq \varepsilon] | B) \leq P([Y_n \in A] | B)$$

$$\leq P([X_n \in A^\varepsilon] \cap [\varrho(X_n, Y_n) < \varepsilon] | B) + P([Y_n \in A] \cap [\varrho(X_n, Y_n) \geq \varepsilon] | B),$$

where $\bar{A} = \Omega \setminus A$, and $A^\varepsilon = \{x: \inf_{y \in A} \varrho(x, y) \leq \varepsilon\}$. Hence, if $A^\varepsilon, (\bar{A})^\varepsilon \in \mathcal{C}_{\mu_B}$, then by the assumption that $\{X_n, n \geq 1\}$ is stable and (3), we get

$$\begin{aligned} \mu_B((\bar{A})^\varepsilon) &= \lim_{n \rightarrow \infty} P([X_n \in (\bar{A})^\varepsilon] | B) \leq \liminf_{n \rightarrow \infty} P([Y_n \in A] | B) \\ &\leq \limsup_{n \rightarrow \infty} P([Y_n \in A] | B) \leq \lim_{n \rightarrow \infty} P([X_n \in A^\varepsilon] | B) = \mu_B(A^\varepsilon). \end{aligned}$$

Letting now $\varepsilon \rightarrow 0$, we obtain

$$\mu_B(A) = \lim_{n \rightarrow \infty} P([Y_n \in A] | B),$$

what completes the proof of Lemma 1.

For the local density $\alpha_A(\omega) = \alpha(A, \omega)$, we have the following

Lemma 2. Let $\alpha(A, \omega)$ be the local density of the stable sequence $\{X_n, n \geq 1\}$ of r.e. Then there is a variant $\lambda(A, \omega)$ of $\alpha(A, \omega)$ such that with probability 1 $\lambda(\cdot, \omega)$ is a probability measure on $(S, \mathcal{B})$ i.e. $P\{\omega: \lambda(A, \omega) \neq \alpha(A, \omega)\} = 0$ for every $A \in \mathcal{B}$.

PROOF. If $A_1 \subset A_2, A_1, A_2 \in \mathcal{B}$, then by the stability of $\{X_n, n \geq 1\}$, we have

$$(4) \quad P\{\omega: \alpha(A_1, \omega) \leq \alpha(A_2, \omega)\} = 1.$$

Moreover, we see that for every $n \geq 1$

$$(5) \quad P\{\omega: \alpha(\bigcup_{i=1}^n A_i, \omega) \neq \sum_{i=1}^n \alpha(A_i, \omega)\} = 0, \quad A_i \cap A_j = \emptyset, \quad i \neq j, \quad A_i \in \mathcal{B},$$

and also that

$$(6) \quad P\{\omega: \alpha(A, \omega) = \sum_{i=1}^{\infty} \alpha(A_i, \omega)\} = 1, \quad \text{where } A_i \cap A_j = \emptyset, \quad i \neq j, \quad A = \bigcup_{i=1}^{\infty} A_i.$$

Let $\{s_i, i \geq 1\}$ be a dense subset of S and $K(s_i, r_i), i \geq 1, l \geq 1$ be a family of all balls such that $r_i \in W$ — the set of rational numbers. One can see, by (4), that there exists a set $T \in \mathcal{A}$ with $P(T) = 0$ such that for every $\omega \notin T$ $K(s_i, r_i) \subset K(s_{i'}, r_{i'})$ implies $\alpha(K(s_i, r_i), \omega) \leq \alpha(K(s_{i'}, r_{i'}), \omega)$ where i, i', l, l' are positive integers. Let $\mathcal{K}$ be the field generated by the finite sums of finite intersections of balls and their complements [4], p. 23. Using (4) and (5) one can conclude that there exists a set $T^0 \in \mathcal{A}, T \subset T^0$ with $P(T^0) = 0$ such that for $\omega \notin T^0, \alpha(\cdot, \omega)$ is a finitely additive set function on $\mathcal{K}$.

Let now for $\omega \notin T^0$

$$\alpha_\varepsilon(A, \omega) = \inf \left\{ \sum_{i=1}^{\infty} \alpha(K_i, \omega), K_i \in \mathcal{K}, d(K_i) < \varepsilon, A \subset \bigcup_{i=1}^{\infty} K_i \right\},$$

where $d(K_i)$ denotes the diameter of K_i . Then

$$(7) \quad \lambda_\varepsilon(A, \omega) = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon(A, \omega)$$

defines an outer metrical measure on the class of all subsets of S and its restriction λ to $\mathcal{B}$ is a probability measure on $(S, \mathcal{B})$ [6].

Put $\mathcal{B}^* = \{A: \alpha(A, \omega) = \lambda(A, \omega) \text{ a.s., } A \in \mathcal{B}\}$.

If $A \in \mathcal{K}$, then we see that

$$\begin{aligned}\alpha_{1/n}(A, \omega) &= \inf \left\{ \sum_{i=1}^{\infty} \alpha(K_i, \omega) : K_i \in \mathcal{K}, d(K_i) < \frac{1}{n}, A \subset \bigcup_{i=1}^{\infty} K_i \right\} = \\ &= \inf \left\{ \sum_{i=1}^{\infty} \alpha(K'_i, \omega) : K'_i \in \mathcal{K}, d(K'_i) < \frac{1}{n}, A = \bigcup_{i=1}^{\infty} K'_i, K'_i \cap K'_j = \emptyset, i \neq j \right\}.\end{aligned}$$

Let $\{K_i^m, i \geq 1, m \geq 1\}$ be a family of sets of $\mathcal{K}$ such that

$$d(K_i^m) < \frac{1}{n}, i \geq 1, n \geq 1, \bigcup_{i=1}^{\infty} K_i^m = A, K_i^m \cap K_j^m = \emptyset, i \neq j, n \geq 1, m \geq 1$$

and

$$\alpha_{1/n}(A, \omega) + \frac{1}{m} > \sum_{i=1}^{\infty} \alpha(K_i^m, \omega), \omega \notin T^0, n \geq 1, m \geq 1.$$

By (6), we have for $m \geq 1$

$$\alpha(A, \omega) = \sum_{i=1}^{\infty} \alpha(K_i^m, \omega), \omega \notin T_m, P(T_m) = 0.$$

Hence, for $n \geq 1$

$$\alpha_{1/n}(A, \omega) = \alpha(A, \omega) \text{ a.s. (i.e. } \omega \notin T^0 \cup \bigcup_{m=1}^{\infty} T_m),$$

and by (7),

$$\lambda(A, \cdot) = \alpha(A, \cdot) \text{ a.s.}$$

as $A \in \mathcal{K}$. Thus $\mathcal{K} \subset \mathcal{B}^*$.

We shall now show that $\mathcal{B}^*$ is a σ -field. Indeed, if $A \in \mathcal{B}^*$, then by (4)

$$\lambda(S \setminus A, \cdot) = \lambda(S, \cdot) - \lambda(A, \cdot) = \alpha(S, \cdot) - \alpha(A, \cdot) = \alpha(S \setminus A, \cdot) \text{ a.s.}$$

If now $A_n \in \mathcal{B}^*, n \geq 1, A_n \subset A_{n+1}$, then

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n, \cdot\right) = \lim_{n \rightarrow \infty} \lambda(A_n, \cdot) = \lim_{n \rightarrow \infty} \alpha(A_n, \cdot) = \alpha\left(\bigcup_{n=1}^{\infty} A_n, \cdot\right) \text{ a.s.}$$

as α satisfies (6). Therefore, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}^*$, which completes the proof that $\mathcal{B}^*$ is the σ -field generated by $\mathcal{K}$, i.e. $\mathcal{B}^* = \mathcal{B}$ [4], p. 27. Thus we have proved that there exists a variant $\lambda(A, \omega)$ of $\alpha(A, \omega)$ such that $\lambda(\cdot, \omega)$ is a probability measure on $(S, \mathcal{B})$ with probability 1.

We now shall prove the following

Theorem 2. Let $(S, \rho), (S, \rho')$ be metric spaces, and $g: S \rightarrow S'$ be a continuous map. If $\{X_n, n \geq 1\}$ is a stable sequence of r.e. with density α_A , then the sequence $\{g(X_n), n \geq 1\}$ is also stable with density given by

$$(8) \quad \beta(A', \omega) = \alpha[g(s) \in A', \omega], \quad A' \in \mathcal{B}'.$$

PROOF. By the assumption for every $B \in \mathcal{A}, P(B) > 0$,

$$(P_B)_{X_n} \Rightarrow \mu_B, \quad n \rightarrow \infty, \quad (\Rightarrow \text{-weakly converges})$$

takes place, where values of $(P_B)_{X_n}$ are defined by $P([X_n \in A] | B)$, $A \in \mathcal{B}$. Using Theorem 5.1 of [2], p. 30, we have

$$(P_B)_{X_n} g^{-1} \Rightarrow \mu_B g^{-1} \quad (\mu_B g^{-1}(A') = \mu_B(A'), A' \in \mathcal{B}'),$$

what proves that $\{g(X_n), n \geq 1\}$ is stable, and moreover,

$$\begin{aligned} \int_B \beta_{A'} dP &= Q_{A'}(B) = \mu_{B'}'(A') P(B) = \mu_B(g^{-1}(A')) P(B) = \\ &= Q_{(g^{-1}(A'))}(B) = \int_B \alpha_{g^{-1}(A')} dP, \end{aligned}$$

where $\beta_{A'}$ is the density of the stable sequence $\{g(X_n), n \geq 1\}$. Hence

$$\beta_{A'} = \alpha_{g^{-1}(A')} \quad \text{a.e.} \quad A' \in \mathcal{B}'.$$

Remark 1. It is easy to see that Theorem 2 is true when $g: S \rightarrow S'$ and $\mu_\Omega(D_g) = 0$, where D_g denotes the set of all discontinuity points of g .

From Theorem 2 we get the following well-known facts:

Corollary 1. *If $\{X_n, n \geq 1\}$ is a mixing sequence of r.e., then $\{g(X_n), n \geq 1\}$ is a mixing sequence.*

Theorem 3. *Let $\{X_n, n \geq 1\}$ be a stable sequence of r.v. with the density $\alpha_x = 0$ or 1 a.e. for every $x \in R$. Then the density $\alpha_A = 0$ or 1 a.e. for every $A \in \mathcal{B}$.*

PROOF. Let $\mathcal{H}$ stand for the class of all sets $H \subset R$ such that

$$\lim_{n \rightarrow \infty} P([X_n \in H] | B) = \frac{1}{P(B)} \int_B \alpha_H dP,$$

and the density $\alpha_H = 0$ or 1 a.e. (of course, $(-\infty, x) \in \mathcal{H}$). We shall prove that $\mathcal{H}$ is a σ -field. By Lemma 2 we can assume and do that the density α_A of $\{X_n, n \geq 1\}$ is a probability measure with respect to A for almost all $\omega \in \Omega$. Therefore,

$$\alpha(\bar{H}, \cdot) = 1 - \alpha(H, \cdot) \quad \text{a.e.,} \quad H \in \mathcal{H},$$

whence $\bar{H} \in \mathcal{H}$.

Since

$$\alpha((a, b), \cdot) = \alpha((-\infty, b), \cdot) - \alpha((-\infty, a), \cdot),$$

it follows that $(a, b) \in \mathcal{H}$.

Let now $H_1, H_2 \in \mathcal{H}$, $H_1 \cap H_2 = \emptyset$. Then

$$0 \leq \alpha(H_1 \cup H_2, \cdot) = \alpha(H_1, \cdot) + \alpha(H_2, \cdot) \leq 1 \quad \text{a.e.,}$$

what proves that $\alpha(H_1 \cup H_2, \cdot) = 0$ or 1 a.e., so $H_1 \cup H_2 \in \mathcal{H}$. Thus we have proved that $\mathcal{H}$ contains a field of the sets of the following form:

$$(9) \quad H = \bigcup_{i=1}^n (a_i, b_i), \quad (a_i, b_i) \cap (a_j, b_j) = \emptyset, \quad i \neq j.$$

Let $H_n \in \mathcal{H}$, $n \geq 1$, $H_1 \subset H_2 \subset \dots \subset H_n \subset \dots$, and $H_0 = \bigcup_{n=1}^{\infty} H_n$. Then

$$\alpha(H_0, \cdot) = \lim_{n \rightarrow \infty} \alpha(H_n, \cdot)$$

which implies that $\alpha(H_0, \cdot) = 0$ or 1 a.e., so $H_0 \in \mathcal{H}$. Therefore, by Theorem A of [4], p. 27, we see that $\mathcal{H}$ is a σ -field, which completes the proof of Theorem 3.

It is known that if a stable sequence $\{X_n, n \geq 1\}$ of r.e. contains a mixing subsequence, then the sequence $\{X_n, n \geq 1\}$ is mixing.

We see that the following facts hold true [12].

Lemma 3. *If a stable sequence $\{X_n, n \geq 1\}$ of r.e. contains a subsequence which converges in probability to a r.e. X ($X_{n_k} \xrightarrow{P} X, k \rightarrow \infty$), then the sequence $\{X_n, n \geq 1\}$ converges in probability to X .*

Remark 2. Let $\{X_n, n \geq 1\}$ be a sequence of r.e. convergent in probability to X . The sequence $\{X_n, n \geq 1\}$ is mixing iff X is a degenerate r.e.

4. Let (S, ρ) , (S', ρ') and (S^*, ρ^*) be nondegenerate polish metric spaces with Borel σ -fields $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}^*$, respectively. By $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{X}^*$ we denote the sets all r.e. defined on $(\Omega, \mathcal{A}, P)$ taking values in S , S' and S^* , respectively.

To prove the next theorem we need the following lemma.

Lemma 4. *Let $\{X_n \in \mathcal{X}; n \geq 1\}$ be a sequence of r.e. such that $X_n \xrightarrow{D} X, n \rightarrow \infty$. If a sequence $\{Y_n \in \mathcal{X}', n \geq 1\}$ of r.e. converges in probability to a r.e. Y , then for every continuous function $g: S \times S' \rightarrow S^*$, we have*

$$\rho^*(g(X_n, Y_n), g(X_n, Y)) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

PROOF. Since $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are tight, then for any given $\varepsilon > 0$ there exist compact sets $K \subset S$ and $K' \subset S'$ such that

$$P([X_n \in K] \cap [Y_n \in K']) > 1 - \varepsilon, \quad n \geq 1.$$

The function g is uniformly continuous on the set $K \times K'$. Thus there exists a $\delta > 0$ such that

$$\rho^*(g(x, y), g(x', y')) < \varepsilon \quad \text{if} \quad \sqrt{\rho^2(x, x') + \rho'^2(y, y')} < \delta.$$

Let $\{D_i, i = 1, 2, \dots, m\}$ be a sequence of sets such that

$$\bigcup_{i=1}^m D_i \supset K', \quad d'(D_i) < \delta, \quad \text{and} \quad D_i \in \mathcal{C}_{P_Y}$$

($d'(D)$ denotes the diameter of D). It is obvious that

$$\begin{aligned} & P(\rho^*(g(X_n, Y_n), g(X_n, Y)) > \varepsilon') \leq \\ & \leq \sum_{i=1}^m P([\rho^*(g(X_n, Y_n), g(X_n, Y)) > \varepsilon'], Y \in D_i, Y_n \in D_i, X_n \in K) + \\ & + \sum_{i=1}^m P([\rho^*(g(X_n, Y_n), g(X_n, Y)) > \varepsilon', Y \notin D_i, Y_n \in D_i, X_n \in K] + \varepsilon). \end{aligned}$$

The first sum equals 0 and

$$\begin{aligned} \lim_{n \rightarrow \infty} P(\varrho^*(g(X_n, Y_n), g(X_n, Y)) > \varepsilon') &\leq \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^m P(Y \notin D_i, Y_n \in D_i) + \varepsilon = \varepsilon, \end{aligned}$$

which completes the proof.

Definition 2. A sequence $\{X_n, n \geq 1\}$ of r.v. is said to be w -star convergent to a r.v. $X(X_n \xrightarrow{w*} X$ as $n \rightarrow \infty$) if for every $D \in \mathcal{A}$

$$\lim_{n \rightarrow \infty} \int_D X_n dP = \int_D X dP.$$

Theorem 4. Let $\{X_n \in \mathcal{X}, n \geq 1\}$ be a stable sequence of r.e. with the density function $\alpha(A, \cdot)$. Suppose that a sequence $\{Y_n \in \mathcal{X}', n \geq 1\}$ of r.e. converges in probability to a r.e. $Y \in \mathcal{X}'$.

If a function $g: S \times S' \rightarrow S^*$ is continuous, then the sequence $\{Z_n \in \mathcal{X}^*, n \geq 1\}$ of r.e. defined by $Z_n = g(X_n, Y_n), n \geq 1$, is stable with the density $\beta(A^*, \cdot)$, given by

$$(10) \quad \beta(A^*, \omega) = \alpha(\{x: g(x, Y(\omega)) \in A^*\}, \omega), \quad A^* \in \mathcal{B}^*.$$

PROOF. Let $Z'_n = g(X_n, Y)$, $n \geq 1$. From Lemmas 4 and 1, we conclude that it is enough to show that the sequence $\{Z'_n, n \geq 1\}$ is stable with the density (10).

Now by results of [1] it is known that a sequence $\{X_n \in \mathcal{X}, n \geq 1\}$ of r.e. is stable iff for every r.e. $V \in \mathcal{X}'$ the sequence $\{(X_n, V), n \geq 1\}$ converges weakly in $(S \times S', \mathcal{B} \times \mathcal{B}')$. Hence, we see that the sequence $\{(X_n, Y), n \geq 1\}$ is stable with the density $\alpha'(A', \cdot)$, $A' \in \mathcal{B} \times \mathcal{B}'$, and Theorem 2 establishes the stability of the sequence $\{Z'_n, n \geq 1\}$.

We see that the measures

$$(11) \quad Q_A^n(D) = P([X_n \in A], D)$$

are absolutely continuous with respect to the measure P , so by the Radon—Nikodym theorem there exist measurable functions $\alpha_n(A, \cdot)$ such that

$$(12) \quad Q_A^n(D) = \int_D \alpha_n(A, \cdot) dP.$$

We can assume and do that $\alpha_n(A, \omega)$ for almost all ω are measures on $(S, \mathcal{B})$. Moreover, by the stability of $\{X_n, n \geq 1\}$, we have for every $D \in \mathcal{A}$

$$(13) \quad \lim_{n \rightarrow \infty} \int_D \alpha_n(A, \cdot) dP = \int_D \alpha(A, \cdot) dP = Q_A(D) = \mu_D(A) P(D), \quad A \in \mathcal{C}_{\mu_D}.$$

Hence

$$\alpha_n(A, \cdot) \xrightarrow{w*} (A, \cdot), \quad n \rightarrow \infty, \quad \text{for } A \in \mathcal{C}_{\mu_D}.$$

Note now that for $Y \in \mathcal{X}'$, $B \in \mathcal{B}'$ and $D \in \mathcal{A}$, $P(D) > 0$, we have by (11) and (12)

$$(14) \quad \begin{aligned} P([X_n \in A], [Y \in B], D) &= \int_{[Y \in B] \cap D} \alpha_n(A, \cdot) dP = \\ &= \int_D I_{[Y \in B]} \alpha_n(A, \cdot) dP = \int_D \mu_{Y(\cdot)}(B) \alpha_n(A, \cdot) dP, \end{aligned}$$

where $\mu_{Y(\cdot)}(B) = I_{[Y \in B]}(\cdot)$.

Now we see also that the measures $Q_A^n(D)$ defined on the space $(\Omega, \mathcal{A})$ by the formula

$$(15) \quad Q_A^n(D) = P([(X_n, Y) \in A'], D), \quad A' \in \mathcal{B} \times \mathcal{B}'$$

are absolutely continuous with respect to the measure P and that there exists a measurable function $\alpha'_n(A', \cdot)$ such that

$$Q_A^n(D) = \int_D \alpha'_n(A', \cdot) dP.$$

In the particular case with $A' = A \times B$, we have

$$Q_{A \times B}^n(D) = \int_D \alpha'_n(A \times B, \cdot) dP.$$

Similarly, as above, we can assume that $\alpha'_n(A', \cdot)$ are measures on $(S \times S', \mathcal{B} \times \mathcal{B}')$, and state that

$$\alpha'_n(A \times B, \cdot) = I_{[Y \in B]}(\cdot) \alpha_n(A, \cdot) = \mu_{Y(\cdot)}(B) \alpha_n(A, \cdot) \quad \text{a.e.}$$

Moreover, we have

$$\alpha'_n(A \times B, \cdot) = \mu_{Y(\cdot)}(B) \alpha_n(A, \cdot) \xrightarrow{w^*} \mu_{Y(\cdot)}(B) \alpha(A, \cdot),$$

and

$$\alpha'_n(A \times B, \cdot) \xrightarrow{w^*} \alpha'(A \times B, \cdot), \quad \text{for } A \in \mathcal{C}_{\mu_n} \text{ and } B \in \mathcal{C}_{P_Y},$$

whence

$$\alpha'(A \times B, \cdot) = \mu_{Y(\cdot)}(B) \alpha(A, \cdot) \quad \text{a.e. } A \in \mathcal{C}_{\mu_n}, \quad B \in \mathcal{C}_{P_Y}.$$

Let now $A'_x = \{y: (x, y) \in A'\}$, and $A'_y = \{x: (x, y) \in A'\}$. Then

$$\begin{aligned} \alpha'(A', \cdot) &= \int_S \alpha'(A'_y \times S, \cdot) \alpha'(S \times dy, \cdot) = \int_S I_{[Y \in S]} \alpha(A'_y, \cdot) \mu_{Y(\cdot)}(dy) \alpha(S, \cdot) = \\ &= \int_S \alpha(A'_y, \cdot) \mu_{Y(\cdot)}(dy) = \alpha(A'_{Y(\cdot)}, \cdot), \end{aligned}$$

what proves that the density function of the sequence $\{(X_n, Y), n \geq 1\}$ is given by

$$(16) \quad \alpha'(A', \omega) = \alpha(A_{Y(\omega)}, \omega) = \alpha([s: (s, Y(\omega)) \in A'], \omega).$$

Therefore, if a function $g: S \times S' \rightarrow S^*$ is continuous, then, by Theorem 2, the sequence $\{Z_n, n \geq 1\}$, $Z_n = g(X_n, Y)$, $n \geq 1$, is stable with the density function

$$\beta(A^*, \omega) = \alpha([s: g(s, Y(\omega)) \in A^*], \omega),$$

which gives the proof that the density function of $\{Z_n \in \mathcal{X}^*, n \geq 1\}$ is given by (10).

The following example shows that the assumption of the stability of the sequence $\{X_n, n \geq 1\}$ cannot be omitted.

Example. Let X be a r.v. having the normal distribution with mean 0 and variance σ^2 . Put

$$X_n = \begin{cases} X, & \text{when } n \text{ is even,} \\ -X, & \text{when } n \text{ is odd.} \end{cases}$$

Setting $Y_n = X$, and $g(x, y) = |x - y|$, we get

$$g(X_n, Y_n) = \begin{cases} 0, & \text{when } n \text{ is even,} \\ 2|X|, & \text{when } n \text{ is odd.} \end{cases}$$

Hence, we conclude that the sequence $\{Z_n, n \geq 1\}$, $Z_n = g(X_n, Y_n)$, $n \geq 1$, does not converge in law.

Theorem 4 allows us to prove the following property of the stable sequence $\{X_n, n \geq 1\}$ of r.e.

Theorem 5. Let $\{X_n, n \geq 1\}$ be a stable sequence of r.e. Then for every r.e. $Y \in \mathcal{X}$ there exists a limit

$$(17) \quad a(Y) = \lim_{n \rightarrow \infty} r(X_n, Y),$$

where $r(X, Y) = \inf \{\varepsilon > 0: P[\varrho(X, Y) > \varepsilon] < \varepsilon\}$.

PROOF. Define the function g by $g(x, y) = \varrho(x, y)$. It is obvious that g is continuous. Thus, by Theorem 4, the sequence $\{g(X_n, Y), n \geq 1\}$ converges weakly to a r.v. Z . Hence, for every $\varepsilon \in \mathcal{C}_{F_Z}$ there exists

$$\lim_{n \rightarrow \infty} P[\varrho(X_n, Y) > \varepsilon] = b(Y, \varepsilon).$$

Let

$$a(Y) = \inf \{\varepsilon > 0: b(Y, \varepsilon) < \varepsilon\}.$$

We now prove that

$$\lim_{n \rightarrow \infty} r(X_n, Y) = \liminf_{n \rightarrow \infty} \{\varepsilon > 0: P[\varrho(X_n, Y) > \varepsilon] < \varepsilon\} = a(Y).$$

For every $\delta > 0$, $\delta \in \mathcal{C}_{F_Z}$, there exists n_0 such that for $n \geq n_0$

$$a(Y) - \delta \leq P[\varrho(X_n, Y) > a(Y) - \delta] \rightarrow b(Y, a(Y) - \delta),$$

and

$$a(Y) + \delta \geq P[\varrho(X_n, Y) > a(Y) + \delta] \rightarrow b(Y, a(Y) + \delta).$$

Hence

$$a(Y) - \delta \leq r(X_n, Y) \leq a(Y) + \delta, \quad n \geq n_0.$$

Letting $\delta \rightarrow 0$, we get

$$\lim_{n \rightarrow \infty} r(X_n, Y) = a(Y),$$

which completes the proof of Theorem 5.

Theorem 5'. Let $\{X_n, n \geq 1\}$ be a stable sequence of r.e. Then for every r.e. $Y \in \mathcal{X}$, there exists a limit

$$a_1(Y) = \lim_{n \rightarrow \infty} r_1(X_n, Y),$$

where $r_1(X, Y) = E \frac{\varrho(X, Y)}{1 + \varrho(X, Y)}$.

PROOF. We see that the sequence $\left\{ \frac{\varrho(X_n, Y)}{1 + \varrho(X_n, Y)}, n \geq 1 \right\}$ is uniformly integrable, and by Theorem 4, it is also stable. Therefore, by Theorem 3 of [5], there exists

$$\lim_{n \rightarrow \infty} r_1(X_n, Y) = a_1(Y).$$

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