

pE Loops

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Abstract. Some properties of finite pE loops, a class of Moufang loops, are investigated. It is proved that : (i) Moufang loops of order p^4 are pE loops. (ii) Moufang loops of order p^5 are pE loops for $p > 3$. If G is a pE loop, then : (iii) G_a , the associator subloop of G , is of exponent p ; G_a is an elementary abelian p -group if $p \neq 3$. (iv) G satisfies Lagrange's Theorem and G has Sylow p -subloops for each p dividing the order of G . (v) $R = \langle R(x, y), L(x, y) | x, y \in G \rangle$, the subgroup of multiplicative group of G is an elementary abelian p -group if $p \neq 3$. (vi) $G_a \subset Z$ for $p > 3$.

Definitions & Notations: A loop G is a Moufang loop if $xy \cdot zx = (x \cdot yz)x$ for all $x, y, z \in G$.

G_a , the associator subloop of G , is generated by all the associators (x, y, z) where $xy \cdot z = (x \cdot yz)(x, y, z)$. G_c , the commutator subloop of G , is generated by all the commutators $[x, y]$ where $xy = yx \cdot [x, y]$. Let the center and nucleus of G be denoted by Z and N . It is known that both are normal in G ; N and Z are groups; Z is abelian and $Z \subset N$. A Moufang loop is a pE loop if $\frac{G}{N}$ is commutative of exponent p , p a prime.

Fundamental Lemma. Let G be a Moufang loop. Then G satisfies all or none of the following identities:

(i) $[(x, y, z), x] = 1$; (ii) $(x, y, [y, z]) = 1$; (iii) $(x, y, z)^{-1} = (x^{-1}, y, z)$; (iv) $(x, y, z)^{-1} = (x^{-1}, y^{-1}, z^{-1})$; (v) $(x, y, z) = (x, zy, z)$; (vi) $(x, y, z) = (x, z, y^{-1})$; (vii) $(x, y, z) = (x, xy, z)$. When these identities hold, then the associator (x, y, z) lies in the centre of the subloop generated by x, y, z ; and the following identities hold for all integers n :

$$(x, y, z) = (y, z, x) = (y, x, z)^{-1}$$

$$(x^n, y, z) = (x, y, z)^n$$

$$[xy, z] = [x, z][[x, z], y][y, z](x, y, z)^3.$$

PROOF. [1, p. 125, Lemma 5.5.]

Remark. A Moufang loop satisfying all the identities of the Fundamental Lemma is called an F loop.

Properties of G_a : If G is a Moufang loop, then $G_a \triangleleft G$, i.e. G_a is normal in G . Also $G_a \subset C_G(N) = \{g | gn = ng \ \forall n \in N\}$.

PROOF. [9, p. 33—34.]

Remark. A loop G is a $2E$ loop if and only if G is an extra loop, [5, p. 190, Theorem 1]. Commutative Moufang loops are $3E$ loops. There exists nonassociative $5E$ loops, [11, p. 408]. Many other pE loops can be constructed by holomorphy theory of loops [10, p. 141]. G/N is commutative implies $G_c \subset N$. Thus $(x, y, [y, z]) = 1$ for all $x, y, z \in G$. So pE loops are F loops. The Fundamental lemma will be applied on pE loops repeatedly without mention. All definitions and notations follow those in [1], unless otherwise stated. All loops are assumed finite.

Theorem 1. *A Moufang loop G of order p^4 is a pE loop; G is a group if $p > 3$.*

PROOF. By [6, p. 397, Theorem 4] and [6, p. 415, Theorem], G is nilpotent. Thus Z is nontrivial. If $|Z| > p$, then $G = \langle Z, x, y \rangle$ for some $x, y \in G$. As a Moufang loop is diassociative, G is a group. But a group is a pE loop. If $|Z| = p$, then $|G/Z| = p^3$. By [8, p. 33, Lemma 1], G/Z is a group. If G/Z is generated by two elements, then G is a group by diassociativity. If G/Z is generated by three elements, then G/Z is an elementary abelian p -group by [4, p. 145]. As $Z \subset N$, G/N is an elementary abelian p -group.

For the second statement of Theorem 1, see [8, p. 33].

Lemma 1. *Let x, y, z be elements of an F loop G . Then*

(i) $zR(x, y) = z(z, x, y)$ where $R(x, y) = R(x)R(y)R(xy)^{-1}$ is an inner mapping of the multiplicative group of G .

(ii) $(xy)\Theta = x\Theta y\Theta(x\Theta, y\Theta, c^{-1})$ where Θ is a pseudoautomorphism of G with companion c .

PROOF. By [3, p. 49, Lemma 1].

Theorem 2. *A nonassociative Moufang loop G of order p^5 is a pE loop for $p \geq 5$.*

PROOF. If $p^2 \nmid |N|$, then $|G/N| \leq p^3$. As G is diassociative, G/N must be generated by three elements. By [4, p. 145], $G/N = C_p \times C_p \times C_p$ where C_p is a cyclic group of order p . As G is nilpotent, $Z \neq 1$. As $Z \subset N$, we can assume $Z = N = C_p$. So, $|G/Z| = p^4$. By Theorem 1, G/Z is a group or $G_a \subset Z$. As G is nonassociative, $G_a \neq 1$. Thus $G_a = Z = N = C_p$. So $(x, y, z)^p = 1$ for all $x, y, z \in G$. Therefore $(x^p, y, z) = 1$ or $x^p \in N$ for all $x \in G$. If $G' = C_p$, then $G_a = G' = N$. So G/N is commutative of exponent p . Suppose $G' \neq C_p$. By [1, p. 98, Theorem 2.2], $G' \subset \phi(G)$, the Frattini subloop of G . As G is diassociative but nonassociative, $|\phi(G)| \leq p^2$. So $|G'| \leq p^2$. Assume $|G'| = p^2$. Then G_2 from the descending central series of G is equal to Z . ($G_2 = 1 \Leftrightarrow G' \subset Z$.) $|G'| = p^2$ implies that $G = \langle x, y, z \rangle$ for some $x, y, z \in G$. Let $u, v, w \in G$. As $[w, [x, y]] \in Z$, $(u, v, [w, [x, y]]) = 1$. By Lemma 1,

$$\begin{aligned} uvR(w, [x, y]) &= uv(uv, w, [x, y]) \\ &= u(u, w, [x, y]) \cdot v(v, w, [x, y]) = uv \cdot (u, w, [x, y])(v, w, [x, y]) \end{aligned}$$

since $G_a \subset Z$.

Thus $(uv, w, [x, y]) = (u, w, [x, y])(v, w, [x, y])$. By repeating the expansion for associators on the R.H.S. we see that $(uv, w, [x, y])$ is a product of associators of the form $(e, d, [x, y])$ where e, d are elements of the set $\{x, y, z\}$. By Funda-

mental Lemma $(e, d, [x, y]) = 1$. Thus $[x, y] \in N$. Similarly $[y, z]$ and $[z, x]$ are in N . This implies $G_c \subset N$. So $G' \subset N$. This is a contradiction.

Theorem 3. *Let G be a pE loop. Then:*

- (i) G_a is an elementary abelian p -group if $p \neq 3$.
- (ii) G_a is a loop of exponent 3 if $p = 3$.
- (iii) G satisfies Lagrange's Theorem (i.e. the order of a subloop divides $|G|$).
- (iv) G contains p -Sylow subloops.

PROOF. (i) G/N is commutative of exponent p implies $G_c \subset N$ and $x^p \in N$ for all $x \in G$. Let $b \in G_a$. Then $a^p \in N$. By the Fundamental lemma, $b^3 \in N$. As $(p, 3) = 1$, $b \in N$. Thus $G_a \subset N$. By the properties of G_a , $G_a \subset Z(N)$, the centre of N . Thus G_a is an abelian group. Now $x^p \in N$ implies that $(x^p, y, z) = (x, y, z)^p = 1$ for all $y, z \in G$. So G_a is an elementary abelian p -group.

(ii) G/N is commutative of exponent 3 implies $G_c \subset N$ and $x^3 \in N$ for all $x \in G$. Thus $(x, y, z)^3 = (x^3, y, z) = 1$, $y, z \in G$. Let $a = (x, y, z)$ and $b = (u, v, w)$. Since $[a, b] \in G_c \subset N$, $[[a, b], a] = 1$ by the Properties of G_a . By [1, p. 122, Lemma 5.1], $(ab)^3 = a^3 b^3 (a, b)^{-3} = (a^3, b)^{-1} = 1$. Thus G_a is of exponent 3.

(iii) Consider $1 \triangleleft G_a \triangleleft G$. G/G_a is a group and G_a is a nilpotent p -loop. Hence both G/G_a and G_a satisfy Lagrange's Theorem. By [2, p. 269, Theorem 6A], G satisfies Lagrange's Theorem.

(iv) By (i) and (ii), G_a is nilpotent. By [Theorem 2, p. 34, 7] G contains p -Sylow subloops.

Theorem 4. $\mathbf{R} = \langle R(x, y), L(x, y) | x, y \in G \rangle$ is an elementary abelian p -group for $p \neq 3$.

PROOF. As $G_c \subset N$, $R(x, y)$ is an automorphism (its companion $[x, y] \in N$). Let $z \in G$.

$$\begin{aligned} \therefore zR^p(x, y) &= z(z, x, y)R^{p-1}(x, y) = \\ &= z(z, x, y)^2R^{p-2}(x, y) \text{ by Fundamental lemma} \\ &\vdots \\ &= z(z, x, y)^p \\ &= z \text{ by Theorem 3.} \end{aligned}$$

$$\therefore R^p(x, y) = I.$$

Now $bR(x, y)R(u, v) = [b(b, x, y)]R(u, v) = b(b, u, v) \cdot (b, x, y)$ since $G_a \subset N$.

$$\begin{aligned} bR(u, v)R(x, y) &= [b(b, u, v)]R(x, y) = b(b, x, y) \cdot (b, u, v) \\ &= b(b, u, v) \cdot (b, x, y) \text{ since } G_a \subset Z(N) \\ &\text{by Properties of } G_a. \end{aligned}$$

$$\therefore R(x, y)R(u, v) = R(u, v)R(x, y).$$

By [1, p. 124, Lemma 5.4], $R(x, y) = L(x^{-1}, y^{-1})$. Thus $\mathbf{R}$ is an elementary abelian p -group.

Remark. For $p=3$, it can be shown that $R(x, y)$ is of order 3. However, it is not known if R is of exponent 3.

Theorem 5. *Let G be a pE loop. Then $G_a \subset Z$ for $p \neq 2, 3$.*

PROOF. Let $u, v, x, y \in G$.

$$\begin{aligned}
 (uv)R(x, y) &= (uv)(uv, x, y) \text{ by Lemma 1} \\
 &= u(u, x, y) \cdot v(v, x, y) \text{ by Lemma 1 and } G_c \subset N. \\
 &= uv(u, x, y)[(u, x, y), v](v, x, y) \quad G_a \subset N. \\
 (uv, x, y) &= (u, x, y)[(u, x, y), v](v, x, y) \\
 (uv, x, y) &= (vu[u, v], x, y) \\
 &= (vu, x, y) \text{ as } G_c \subset N. \\
 &= (v, x, y)[(v, x, y), u](u, x, y)
 \end{aligned}$$

$[(u, x, y), v] = [(v, x, y), u]$ by Properties of G_a and $G_c \subset N$. Thus $[(u, x, y), v]$ is symmetric in u, v . By Fundamental lemma, $[(u, x, y), v]$ is skew-symmetric in u, x, y and in x, y, z . Hence $[(u, x, y), v]$ is skew-symmetric in w, x . Therefore $[(u, x, y), v] = [(u, x, y), v]^{-1}$. So $[(u, x, y), v]^2 = 1$. $[(u, x, y), v], (u, x, y) = 1$ by Properties of G_a . By [1, p. 122, Lemma 5.1], $[(u, x, y), v]^p = [(u, x, y)^p, v] = 1$. Therefore $[(u, x, y), v] = 1$ for all $u, v, x, y \in G$.

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