

On additive arithmetical functions with values in topological groups II

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1. Let G be an additively written Abelian topological group with the translation invariant metric ϱ . A mapping $\varphi: \mathbf{N} \rightarrow G$ is called to be a completely additive function, if

$$(1.1) \quad \varphi(mn) = \varphi(m) + \varphi(n) \quad \forall m, n \in \mathbf{N}$$

holds.

Continuing our work [1] we shall consider completely additive functions under the condition

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{\varrho(\varphi(n), \varphi(n+1))}{n} < \infty.$$

A complete characterization of completely additive functions subject to (1.2) for $G = \mathbf{R}/\mathbf{Z}$ was given by the second named author in a series of papers [2].

Let Q_x , resp. R_x be the multiplicative groups of positive rationals and the positive reals. We can extend the domain of φ to Q_x by the relation

$$\varphi\left(\frac{m}{n}\right) := \varphi(m) - \varphi(n),$$

uniquely. Then φ satisfies the relation

$$\varphi(rs) = \varphi(r) + \varphi(s) \quad \forall r, s \in Q_x,$$

so $\varphi: Q_x \rightarrow G$ is a homomorphism. We shall say that φ is continuous at the point 1, if $r_v \in Q_x$, $r_v \rightarrow 1$ implies that $\varphi(r_v) \rightarrow 0$. In [1] we proved the next

Lemma 1. *Let G be an additively written closed Abelian topological group, $\varphi: Q_x \rightarrow G$ be a homomorphism that is continuous at the point 1. Then its domain can be extended onto R_x by the relation*

$$\varphi(\alpha) := \lim_{\substack{r_v \rightarrow \alpha \\ r_v \in Q_x}} \varphi(r_v) \quad (\alpha \in R_x)$$

uniquely. The so obtained mapping $\varphi: R_x \rightarrow G$ is a continuous homomorphism, consequently

$$(1.3) \quad \varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta) \quad (\forall \alpha, \beta \in R_x).$$

Theorem. *If φ satisfies (1.1), (1.2) then it is a continuous $\mathbf{R}_x \rightarrow G$ homomorphism.*

2. PROOF OF THEOREM. The proof is based on the following

Lemma 2. *Let $p, q \in \mathbf{N}$ be coprime integers, $q < p^2$; $M, N \in \mathbf{N}$. Let (k_t, l_t) be such a sequence in $\mathbf{N}^2$ for which*

$$r_t := \frac{p^{k_t}}{q^{l_t}} \rightarrow \frac{M}{N}$$

holds. Then

$$\varphi(r_t) \rightarrow \varphi(M/N). \quad \blacksquare$$

First we deduce our theorem from this lemma. Let $r_j = N_j \cdot M_j^{-1} \in \mathcal{Q}_x$, $r_j \rightarrow 1$. Let p, q be fixed integers satisfying the conditions stated in Lemma 2. Then for every r_j there exists a suitable sequence $(a_v, b_v) \in \mathbf{N}^2$ such that

$$p^{a_v} \cdot p^{-b_v} \rightarrow r_j \quad (v \rightarrow \infty),$$

and so by Lemma 2

$$\varphi(p^{a_v} q^{-b_v}) \rightarrow \varphi(r_j) \quad (v \rightarrow \infty).$$

Consequently there exists such a sequence $(k_j, l_j) \in \mathbf{N}^2$ for which

$$p^{k_j} q^{-l_j} - r_j \rightarrow 0, \quad \varphi(p^{k_j} q^{-l_j}) - \varphi(r_j) \rightarrow 0.$$

Observing now that $p^{k_j} q^{-l_j} \rightarrow 1$, from Lemma 2 ($M=N=1$) we get that

$$\varphi(p^{k_j} q^{-l_j}) \rightarrow \varphi(1) = 0.$$

Consequently $\varphi(r_j) \rightarrow 0$, the conditions in Lemma 1 are satisfied, and so the Theorem is true.

Let us prove finally Lemma 2.

Let K be a large constant,

$$f(n) = \max_{a \in [-K, K]} \varrho(\varphi(n), \varphi(n+a)).$$

From (1.2) we have

$$\sum_{n \geq 1} \frac{f(n)}{n} < \infty.$$

Let

$$t(y) = \sum_{n \equiv y} \frac{f(n)}{n},$$

furthermore

$$A_m := [Np^m, Np^{m+1}); \quad B_m := [Mq^m, Mq^{m+1}).$$

Let $H_1 \in \mathbf{N}$ be a large constant, $(k, l) \in \mathbf{N}^2$ be such that

$$(2.1) \quad \left| \frac{Np^k}{Mq^l} - 1 \right| < \frac{1}{p^{H_1}},$$

and $H_2 \in \mathbf{N}$ be defined so that $B_{l+H_2} \subseteq A_{k+H_1}$. For an $n \in B_{l+H_2}$ let

$$n_j = n_j(n) = \left\lfloor \frac{n}{p^j} \right\rfloor, \quad m_j = m_j(n) = \left\lfloor \frac{n}{q^j} \right\rfloor \quad (j = 0, 1, 2, \dots).$$

It is clear that $n_j \in A_{k+H_1-j}$, $m_j \in A_{l+H_2-j}$, furthermore each n_j occurs for at most p^j of n , and each m_j occurs for at most q^j of n . Furthermore

$$n_j = pn_{j+1} + b_j, \quad m_j = qm_{j+1} + c_j, \quad 0 \leq b_j < p, \quad 0 \leq c_j < q.$$

Hence we get that

$$\varrho(\varphi(n_j), \varphi(pn_{j+1})) \leq f(n_j), \quad \varrho(\varphi(m_j), \varphi(qm_{j+1})) \leq f(m_j),$$

assuming that $K > p(> q)$.

We have

$$\varrho(\varphi(n), \varphi(p^k n_k(n))) \leq \sum_{s=0}^{k-1} \varrho(\varphi(p^s n_s(n)), \varphi(p^{s+1} n_{s+1}(n))) \leq \sum_{s=0}^{k-1} f(n_s(n)),$$

and similarly that

$$\varrho(\varphi(n), \varphi(q^l m_l(n))) \leq \sum_{s=0}^{l-1} f(m_s(n)).$$

Consequently

$$\varrho(\varphi(p^k n_k(n)), \varphi(q^l m_l(n))) \leq \sum_{s=0}^{k-1} f(n_s(n)) + \sum_{s=0}^{l-1} f(m_s(n)),$$

and so

$$(2.2) \quad \sum_{n \in B_{l+H_2}} \frac{1}{n} \varrho(\varphi(p^k n_k(n)), \varphi(q^l m_l(n))) \leq t(p^{H_1-1}) + t(q^{H_2-1}).$$

We have, by (2.1)

$$(2.3) \quad \begin{aligned} |Mn_k(n) - Nm_l(n)| &\leq n \left| \frac{M}{p^k} - \frac{N}{q^l} \right| + (M+N) \leq \frac{nM}{p^k} \left| 1 - \frac{Np^k}{Mq^l} \right| + M+N \leq \\ &\leq \frac{p^{k+H_1+k} \cdot M}{p^k} \cdot \frac{1}{p^{H_1}} + (M+N) \leq 2M+N. \end{aligned}$$

By using the translation invariant property of ϱ , we get that

$$\begin{aligned} \varrho\left(\varphi\left(\frac{p^k}{M}\right), \varphi\left(\frac{q^l}{N}\right)\right) &= \varrho\left(\varphi(p^k n_k(n)), \varphi\left(\frac{q^l}{N} Mn_k(n)\right)\right) \leq \\ &\leq \varrho(\varphi(p^k n_k(n)), \varphi(q^l m_l(n))) + \varrho(\varphi(Mn_k(n)), \varphi(Nm_l(n))). \end{aligned}$$

Hence, by (2.2) we deduce that

$$(2.4) \quad \begin{aligned} \varrho\left(\varphi\left(\frac{p^k}{M}\right), \varphi\left(\frac{q^l}{N}\right)\right) \cdot \sum_{n \in B_{l+H_2}} \frac{1}{n} &\leq t(p^{H_1-1}) + t(q^{H_2-1}) + \\ &+ \sum_{n \in B_{l+H_2}} \varrho(\varphi(Mn_k(n)), \varphi(Nm_l(n))) \frac{1}{n}. \end{aligned}$$

The sum on the right hand side is majorated by $\ll t(p^{H_1})$. Since

$$1 \leq \sum_{n \in B_{l+H_2}} \frac{1}{n} < q,$$

we get that

$$\varrho \left(\varphi \left(\frac{p^k N}{q^l M} \right), 0 \right) \ll t(p^{H_1-1}) + t(q^{H_2-1}).$$

Let now $(k, l) = (k_t, l_t)$. Then (2.1) is satisfied by a suitable sequence $H_1 = H_1(t) \rightarrow \infty$. This implies that

$$\varphi \left(\frac{p^{k_t} N}{q^{l_t} M} \right) \rightarrow 0,$$

and so the proof of Lemma 2 is finished.

References

- [1] Z. DARÓCZY and I. KÁTAI, On additive arithmetical functions with values in topological groups I *Publ. Math. (Debrecen)* **32** (1985),
- [2] I. KÁTAI, Multiplicative functions with regularity properties I., II., III. *Acta Math. Hung.* **42** (1983), 295—308, **43** (1984), 105—130, **43** (1984), 259—272.

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