# Operational calculus on a subset of $R^n$

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*Abstract*

The space of distributions on $[0, a_1)\times\ldots\times[0, a_n)$ is characterized as the algebra of operators (on a class of test functions) which commute with convolution. With convolution identified with multiplication, some operational formulas are developed.

If $X$ is in $R^n$, then $x_i$ will be understood to denote the $i$-th component of $X$. Thus, $X=(x_1, x_2, \ldots, x_n)$. For $0<a_i\leqq\infty$, define $L$ to be the set of locally integrable complex-valued functions of $n$ real variables on the subset $J=\prod_{i=1}^{n}[0, a_i)$ of $R^n$. These functions are extended to be zero for any $x_i<0$.

*Definition.* For $f$ and $g$ in $L$ we define the convolution

$$f*g(X)=\int_0^{x_n}\ldots\int_0^{x_1} f(X-U)g(U)\,du_1\ldots du_n \quad (X \text{ in } J). \tag{1}$$

The space $L$ is closed under this operation, which is commutative and associative.

Let $Q$ be the subset of $L$ consisting of those functions which are infinitely differentiable and which, along with all partial derivatives, vanish on $\{X\in J \mid \text{some } x_i=0\}$. A mapping $A$ from $Q$ into $Q$ is said to be *perfect* if $A(p*q)=Ap*q$ for all $p$ and $q$ in $Q$. If we define the product of two operators to be the composition of the operators, then the set $P$ of all perfect operators is an algebra.

For $i=1, \ldots, n$, we denote by $f_i$ the partial derivative of $f$ with respect to the $i$-th variable and by $D_i$ the perfect operator which maps $q$ into $q_i$.

Denote by $\mathscr{D}'_+$ the space of all distributions on $\prod_{i=1}^{n}(-\infty, a_i)$ having support in $J$. The space $\mathscr{D}'_+$ can be viewed as the dual of the space $\mathscr{D}_-$ of infinitely differentiable functions having support in some subset $\prod_{i=1}^{n}(-\infty, c_i]$ of $\prod_{i=1}^{n}(-\infty, a_i)$.

*Definition.* For $F$ and $G$ in $\mathscr{D}'_+$ we define

$$\langle F*G, \varphi\rangle=\langle F(X), \langle G(U), \varphi(U+X)\rangle\rangle \quad (\varphi \text{ in } \mathscr{D}_-). \tag{2}$$

The space $\mathscr{D}'_+$ is closed under this operation. For $f$ and $g$ in $L$, the definitions of $f*g$ given in (1) and in (2) agree. For each $F$ in $\mathscr{D}'_+$ and each $q$ in $Q$ the distribution $F*q$ belongs to $Q$ and is given by the equation

$$F*q(X)=\langle F(U), q(X-U)\rangle$$

(cf. [4, Theorem 27.5]).

*Definition.* For $F$ in $\mathscr{D}'_+$ we define $\{F\}q(X)=F*q(X)$ for all $q$ in $Q$.

By the associativity of convolution, $\{F\}$ is a perfect operator and $\{F*G\}=\{F\}\{G\}$ for all $F$ and $G$ in $\mathscr{D}'_+$.

For each $X$ in $R^n$ we denote by $X^i$ the vector $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. For each $i$ we denote by $K_i$ the subset $\{X_i|X\in J\}$ of $R^{n-1}$ and by $L_i$ the set of locally integrable functions on $K_i$. These functions are extended to be zero for any $x_i<0$.

*Definition.* If $h$ belongs to $L_1$, we define

$$\{h\}^1 q(X) = \int_{K_1}\!\!\dots\int h(X^1-U^1)q(x_1, u_2, \dots, u_n)\,du_2\dots du_n$$

for all $X$ in $J$ and $q$ in $Q$. If $h$ belongs to $L_i$, where $2\leqq i\leqq n$, we define $\{h\}^i$ similarly.

**Theorem.** *If $h$ belongs to $L_i$ then $\{h\}^i$ is a perfect operator.*

Proof. We give the proof for $i=1$. If we define $f(X)=h(X^1)$, then

$$\{f\}D_1q(X) = \int_0^{x_n}\dots\int_0^{x_1} f(X-U)q_1(U)\,du\dots du_n =$$

$$= \int_0^{x_n}\dots\int_0^{x_1} h(X^1-U^1)q_1(U)\,du_1\dots du_n =$$

$$= \int_0^{x_n}\dots\int_0^{x_2} h(X^1-U^1)\Big[\int_0^{x_1} q_1(U)\,du_1\Big]\,du_2\dots du_n =$$

$$= \int_0^{x_n}\dots\int_0^{x_2} h(X^1-U^1)q(x_1, u_2, \dots, u_n)\,du_2\dots du_n$$

for all $X$ in $J$ and all $q$ in $Q$. So, $\{h\}^1=D_1\{f\}$; it follows that $\{h\}^1$ is a perfect operator.

**Theorem.** *If $f$ and $f_i$ are in $L$, then*

$$\{f_i\} = D_i\{f\}-\{f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)\}^i$$

*for* $i=1, \dots, n$.

Proof. We give the proof for $i=1$. For each $q$ in $Q$ we can integrate by parts to obtain

$$\{f_1\}q(X) = \int_0^{x_n}\dots\int_0^{x_1} f_1(U)q(X-U)\,du_1\dots du_n =$$

$$= -\int_0^{x_n}\dots\int_0^{x_2} f(0, u_2, \dots, u_n)q(x_1, x_2-u_2, \dots, x_n-u_n)\,du_2\dots du_n +$$

$$+\int_0^{x_n}\dots\int_0^{x_1} f(U)q_1(X-U)\,du_1\dots du_n = -\{f(0, x_2, \dots, x_n)\}^1q(X)+\{f\}D_1q(X) =$$

$$= [D_1\{f\}-\{f(0, x_2, \dots, x_n)\}^1]q(X).$$

We say that a perfect operator $A$ is *invertible* if there is a perfect operator $B$ such that $AB$ is the identity. In this case we write $B=A^{-1}$. As in [3], the perfect operator $D_1+\ldots+D_n$ is invertible; its inverse is given by the equation

$$(D_1+\ldots+D_n)^{-1}q(X)=\int\limits_0^\infty q(x_1-t,\ \ldots,\ x_n-t)\,dt\quad (X\ \text{in}\ J)$$

and the equation

$$(D_1+\ldots+D_n)^{-1}\{f\}=\Big\{\int\limits_0^\infty f(x_1-t,\ \ldots,\ x_n-t)\,dt\Big\}$$

holds for all $f$ in $L$.

We conclude by showing that the space $\mathscr{D}'_+$ is algebraically isomorphic to the space $P$ of perfect operators.

**Theorem.** *The mapping* $F\mapsto\{F\}$ *from* $\mathscr{D}'_+$ *into* $P$ *is linear and one-to-one.*

Proof. The linearity follows from the bilinearity of convolution. Suppose $\{F\}=0$. For $\varphi\in\mathscr{D}_-$ there exist $x_i<a_i$ such that $\varphi(U)=0$ if some $u_i\geqq x_i$. If we define $p(U)=\varphi(X-U)$, then $p\in Q$ and

$$\langle F,\varphi\rangle=F*p(X)=\{F\}p(X)=0.$$

Since $\varphi$ in $\mathscr{D}_-$ was arbitrary, it follows that $F=0$.

**Corollary.** *If $f$ and $g$ are in $L$ and* $\{f\}=\{g\}$, *then* $f=g$ *almost everywhere.*

**Theorem.** *The mapping* $F\mapsto\{F\}$ *is a linear bijection of* $\mathscr{D}'_+$ *onto* $P$.

Proof. It only remains to prove the surjectivity. Let $A$ be a perfect operator. Let $q_1,q_2,q_3,\ldots$ be a "delta sequence" in $Q$. For any $\varphi$ in $\mathscr{D}_-$ there exist, as before, $x_i<a_i$ such that $\varphi(U)=0$ if some $u_i\geqq x_i$. If we define $p(U)=\varphi(X-U)$, then $p\in Q$ and

$$Ap(X)=\lim_{n\to\infty}Aq_n*p(X)=\lim_{n\to\infty}\langle Aq_n(U),\ p(X-U)\rangle=\lim_{n\to\infty}\langle Aq_n(U),\ \varphi(U)\rangle.$$

Since the sequence $\langle Aq_n,\varphi\rangle$ converges for all $\varphi$ in $\mathscr{D}_-$, we infer from the sequential completeness of $\mathscr{D}'_+$ the existence of $F$ in $\mathscr{D}'_+$ such that

$$\langle F,\varphi\rangle=\lim_{n\to\infty}\langle Aq_n,\varphi\rangle\quad(\text{all}\ \varphi\ \text{in}\ \mathscr{D}_-).$$

Now, for any $q$ in $Q$, we have

$$\{F\}q(X)=\langle F(U),\ q(X-U)\rangle=\lim_{n\to\infty}\langle Aq_n(U),\ q(X-U)\rangle=$$

$$=\lim_{n\to\infty}Aq_n*q(X)=\lim_{n\to\infty}q_n*Aq(X)=Aq(X).$$

Thus, $A=\{F\}$.

## Bibliography

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*(Received Marc. 18, 1985.)*