

Operational calculus in algebras

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In this paper on the basis of *Bittner operational calculus*

$$CO(L^0, L^1, S, T(q), s(q), Q)$$

— see [1], [2] we introduce a new operational calculus $CO(L^0, L^1, S_p, T_p(q), s_p(q), Q)$ and the multiplication $\circ$ such that the derivative S_p , and the limit condition $s_p(q)$ satisfy conditions (1) and (2).

Let an operational calculus $CO(L^0, L^1, S, T(q), s(q), Q)$ be given, where $L^1 \subset L^0$, L^1, L^0 are commutative algebras with unity 1, and with the multiplication $\cdot$, such that for $f, g \in L^1$

$$(1) \quad S(f \cdot g) = (Sf) \cdot g + f \cdot (Sg),$$

$$(2) \quad s(q)(f \cdot g) = (s(q)f) \cdot (s(q)g).$$

Theorem 1. *If there exists a solution $u \in \text{Inv}$ of the abstract differential equation*

$$(3) \quad Su = pu$$

with the condition

$$(4) \quad s(q)u = u_0, \quad u \in L^1, \quad p \in L^0, \quad u_0 \in \text{Ker } S,$$

then abstract differential equation (3) with condition (4) has only one solution.

PROOF. If equation (3) with condition (4) had two solutions u, v we would get

$$Su = pu, \quad Sv = pv$$

and

$$s(q)u = u_0, \quad s(q)v = u_0$$

i.e.

$$S(u - v) = p(u - v), \quad s(q)(u - v) = 0$$

because operations S and $s(q)$ are linear operations. On the basis of theorems with [8] $u - v = 0$, then $u = v$. (Theorem 1 is also true when multiplication $\cdot$ is non-commutative.)

Definition 1. We will say that there exists an element $u \stackrel{\text{def}}{=} E_1^{T(q)p}$ if and only if $E_1^{T(q)p}$ is a solution of the abstract differential equation

$$(5) \quad Su = pu$$

with condition

$$(6) \quad s(q)u = \mathbf{1}, \quad \text{where } u \in L^1, \quad p \in L^0 \quad (\mathbf{1} \in \text{Ker } S - \text{see [7]})$$

and $E_1^{T(q)p} \in \text{Inv}$.

Corollary 1. *If there exists an element $E_1^{T(q)p}$, then formula*

$$(7) \quad E_1^{T(q)p} E_1^{-T(q)p} = \mathbf{1}$$

is true

PROOF. Let $v = \mathbf{1} - E_1^{T(q)p} E_1^{-T(q)p}$. Finding Sv and $s(q)v$ we will get $Sv=0$, $s(q)v=0$. From the last two facts it follows that $v=0$, i.e. formula (7) is true.

Theorem 2. *If there exists an element $E_1^{T(q)p}$ then three operations:*

$$(8) \quad S_p u \stackrel{\text{df}}{=} Su + pu,$$

$$(9) \quad T_p(q)f \stackrel{\text{df}}{=} [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p},$$

$$(10) \quad s_p(q)u \stackrel{\text{df}}{=} (s(q)u) \cdot E_1^{-T(q)p}$$

satisfy axioms of operational calculus, where $u \in L^1$, $f \in L^0$. Operation S_p is a derivative, operation $T_p(q)$ is an integral, operation $s_p(q)$ is a limit condition.

PROOF. Operations S_p , $T_p(q)$, $s_p(q)$ are linear operations. Applying the axioms of operational calculus $CO(L^0, L^1, S, T(q), s(q), Q)$ and theorems about derivative, integral and limit condition in algebras (see [3]) we will get

$$\begin{aligned} S_p T_p(q)f &= S_p \{ [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} \} = \\ &= S \{ [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} \} + \\ &+ p \cdot [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = \\ &= f \cdot E_1^{T(q)p} \cdot E_1^{-T(q)p} - p \cdot [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} + \\ &+ p \cdot [T(q)(f \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = f, \end{aligned}$$

$$\text{i.e.} \quad S_p T_p(q)f = f \quad \text{for } f \in L^0.$$

$$\begin{aligned} T_p(q)S_p u &= \{ T(q)[(Su + pu) \cdot E_1^{T(q)p}] \} \cdot E_1^{-T(q)p} = \\ &= \{ T(q)[(Su) \cdot E_1^{T(q)p}] \} \cdot E_1^{-T(q)p} + \\ &+ [T(q)(p \cdot u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = \\ &= u \cdot E_1^{T(q)p} \cdot E_1^{-T(q)p} - [T(q)(p \cdot u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} - \\ &- [s(q)(u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} + \\ &+ [T(q)(p \cdot u \cdot E_1^{T(q)p})] \cdot E_1^{-T(q)p} = \\ &= u - (s(q)u) \cdot E_1^{-T(q)p}, \end{aligned}$$

$$\text{i.e.} \quad T_p(q)S_p u = u - s_p(q)u \quad \text{for } u \in L^1,$$

so operations S_p , $T_p(q)$, $s_p(q)$ satisfy the axioms of Bittner operational calculus.

Corollary 2. If L^1 , L^0 are non-commutative algebras then theorem 2 is true for $p=\alpha 1$, $\alpha \in R$.

Corollary 3. For a given derivative S_p , integral $T_p(q)$ and limit condition $s_p(q)$ the operation

$$\tilde{S}_p u \stackrel{\text{df}}{=} a \cdot (S_p u), \quad a \in L^0, \quad a \in \text{Inv}, \quad u \in L^1$$

is a derivative, the operation

$$\tilde{T}_p(q) f \stackrel{\text{df}}{=} T_p(q) (a^{-1} \cdot f), \quad f \in L^0$$

is an integral, the operation

$$\tilde{s}_p(q) u \stackrel{\text{df}}{=} s_p(q) u, \quad u \in L^1$$

is a limit condition (compare also [4]).

Theorem 3. If $a_1, a_2 \in L^0$, $a_1 \in \text{Inv}$ and if there exists an element $E_1^{T(q)(a_1^{-1} a_2)}$ then the abstract differential equation

$$(11) \quad a_1 \cdot S u + a_2 \cdot u = f$$

with condition

$$(12) \quad s(q) u = u_0, \quad \text{where } u \in L^1, \quad f \in L^0, \quad u_0 \in \text{Ker } S$$

has only one solution defined by formula

$$(13) \quad u = [T(q)(a_1^{-1} \cdot f \cdot E_1^{T(q)(a_1^{-1} a_2)})] \cdot E_1^{-T(q)(a_1^{-1} a_2)} + u_0 \cdot E_1^{-T(q)(a_1^{-1} a_2)}.$$

PROOF. The proof of the theorem follows directly from theorem 2 and corollary 3, and from the theorems of the operational calculus.

Definition 2. If there exists an element $E_1^{T(q)p}$ then for the elements $x, y \in L^0$ we will define the multiplication $x \circ y$ by the formula

$$(14) \quad x \circ y \stackrel{\text{df}}{=} E_1^{T(q)p} \cdot x \cdot y.$$

Properties of the multiplication $\circ$.

1. For the multiplication $\circ$ unity 1_0 is defined by the formula

$$(15) \quad 1_0 = E_1^{-T(q)p}.$$

2. Element $x \in L^0$ has an inverse x^{-1} for the multiplication $\cdot$ if and only if element $x \in L^0$ has an inverse $x^{-1} \circ$ for the multiplication $\circ$.

$$(16) \quad x^{-1} \circ = E_1^{-T(q)p} \cdot E_1^{-T(q)p} \cdot x^{-1}.$$

3. The multiplication $\circ$ satisfies condition (1) for the derivative S_p and condition (2) for the limit condition $s_p(q)$. (Condition (2) usually is not satisfied for $s_p(q_1)$, $q_1 \neq q$.)

4. If the multiplication is defined by formula (14) then operations S_p , $T_p(q)$, $s_p(q)$ satisfy the theorems from chapter II. paragraph 1 of the paper [3].

Examples.

A. Using the operational calculus

$$CO\left(C^0((a, b), R), C^1((a, b), R), \frac{d}{dt}, \int_{t_0}^t, \Big|_{t=t_0}, R\right)$$

we may define the derivative S_p , the integral $T_p(t_0)$ and the limit condition $s_p(t_0)$ by the following formulas

$$S_p\{u(t)\} \stackrel{\text{df}}{=} \left\{ \frac{du(t)}{dt} + p(t)u(t) \right\},$$

$$T_p(t_0)\{f(t)\} \stackrel{\text{df}}{=} \left\{ e^{-\int_{t_0}^t p(\tau)d\tau} \int_{t_0}^t f(\tau) e^{\int_{t_0}^{\tau} p(\xi)d\xi} d\tau \right\} \quad (\text{see [6]})$$

$$s_p(t_0)\{u(t)\} \stackrel{\text{df}}{=} \left\{ u(t_0) e^{-\int_{t_0}^t p(\tau)d\tau} \right\},$$

where $u = \{u(t)\} \in C^1((a, b), R)$, $f = \{f(t)\}$, $p = \{p(t)\} \in C^0((a, b), R)$.

The multiplication $\circ$ defined by formula (14) has the following form

$$x \circ y = \{x(t)\} \circ \{y(t)\} \stackrel{\text{df}}{=} \left\{ e^{\int_{t_0}^t p(\tau)d\tau} x(t)y(t) \right\}, \quad x, y \in C^0((a, b), R)$$

B. In case of the operational calculus with the directional derivative

$$S\{u(x_1, x_2, \dots, x_n)\} \stackrel{\text{df}}{=} \left\{ \sum_{i=1}^n b_i \frac{\partial u(x_1, \dots, x_n)}{\partial x_i} \right\},$$

the integral

$$\begin{aligned} & T(x_n^0)\{f(x_1, x_2, \dots, x_n)\} \stackrel{\text{df}}{=} \\ & \stackrel{\text{df}}{=} \left\{ \frac{1}{b_n} \int_{x_n^0}^{x_n} f\left(x_1 - \frac{b_1}{b_n}(x_n - \tau), x_2 - \frac{b_2}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\right) d\tau \right\} \end{aligned}$$

and the limit conditions

$$\begin{aligned} & s(x_n^0)\{u(x_1, x_2, \dots, x_n)\} \stackrel{\text{df}}{=} \\ & \stackrel{\text{df}}{=} \left\{ u\left(x_1 - \frac{b_1}{b_n}(x_n - x_n^0), x_2 - \frac{b_2}{b_n}(x_n - x_n^0), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - x_n^0), x_n^0\right) \right\} \end{aligned}$$

where $u \in L^1 \stackrel{\text{df}}{=} C^2(R^{n-1} \times \langle x_n^1, x_n^2 \rangle, R)$, $f \in L^0 \stackrel{\text{df}}{=} C^1(R^{n-1} \times \langle x_n^1, x_n^2 \rangle, R)$,

$$x_n^0 \in \langle x_n^1, x_n^2 \rangle, b_i \in R \quad \text{for } i = 1, 2, \dots, n, b_n \neq 0 \quad (\text{see [4]})$$

the multiplication $\circ$ defined by formula (14) has the following form

$$\begin{aligned} x \circ y &= \{x(x_1, x_2, \dots, x_n)\} \circ \{y(x_1, x_2, \dots, x_n)\} \stackrel{\text{df}}{=} \\ &\stackrel{\text{df}}{=} \left\{ e^{\frac{1}{b_n} \int_{x_n^0}^{x_n} p\left(x_1 - \frac{b_1}{b_n}(x_n - \tau), x_2 - \frac{b_2}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\right) d\tau} \right. \\ &\quad \cdot y(x_1, x_2, \dots, x_n) \cdot \left. x(x_1, x_2, \dots, x_n) \right\}. \end{aligned}$$

The derivative S_p , the integral $T_p(x_n^0)$ and the limit condition $s_p(x_n^0)$ are defined by the formulas

$$\begin{aligned} S_p\{u(x_1, x_2, \dots, x_n)\} &\stackrel{\text{df}}{=} \\ &\stackrel{\text{df}}{=} \left\{ \sum_{i=1}^n b_i \frac{\partial u(x_1, x_2, \dots, x_n)}{\partial x_i} + p(x_1, x_2, \dots, x_n) u(x_1, x_2, \dots, x_n) \right\}, \\ T_p(x_n^0)\{f(x_1, x_2, \dots, x_n)\} &\stackrel{\text{df}}{=} \\ &\stackrel{\text{df}}{=} \left\{ e^{-\frac{1}{b_n} \int_{x_n^0}^{x_n} p\left(x_1 - \frac{b_1}{b_n}(x_n - \tau), x_2 - \frac{b_2}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\right) d\tau} \right. \\ &\quad \cdot \frac{1}{b_n} \int_{x_n^0}^{x_n} f\left(x_1 - \frac{b_1}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\right) \cdot \\ &\quad \cdot \left. e^{\frac{1}{b_n} \int_{x_n^0}^{\tau} p\left(x_1 - \frac{b_1}{b_n}(x_n - \xi), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \xi), \xi\right) d\xi} d\tau \right\}, \\ s_p(x_n^0)\{u(x_1, x_2, \dots, x_n)\} &\stackrel{\text{df}}{=} \\ &\stackrel{\text{df}}{=} \left\{ u\left(x_1 - \frac{b_1}{b_n}(x_n - x_n^0), x_2 - \frac{b_2}{b_n}(x_n - x_n^0), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - x_n^0), x_n^0\right) \cdot \right. \\ &\quad \cdot \left. e^{-\frac{1}{b_n} \int_{x_n^0}^{x_n} p\left(x_1 - \frac{b_1}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\right) d\tau} \right\} \end{aligned}$$

where $u \in L^1$, $f, p \in L^0$.

C. Into the space $C(N)$ of real sequences $a = \{a_k\}$ let us introduce the derivative $S = P$ according to the formula

$$P\{a_k\} \stackrel{\text{df}}{=} \{a_{k+1}\}.$$

Introducing to the space $C(N)$ the multiplication of the sequences $a = \{a_k\}$, $b = \{b_k\}$ according to the formula

$$a * b \stackrel{\text{df}}{=} \left\{ \sum_{i=0}^k \binom{k}{i} a_i b_{k-i} \right\}$$

we may prove that condition

$$S(a * b) = (Sa) * b + a * (Sb)$$

is satisfied (see [2]).

The limit condition s corresponding to the derivative $S=P$ has the form

$$s\{a_k\} \stackrel{\text{df}}{=} \begin{cases} a_0 & \text{for } k=0 \\ 0 & \text{for } k=1, 2, \dots \end{cases} \quad (\text{see [2]}).$$

On the basis of theorem 2 we can define the derivative S_p , the integral T_p and the limit condition s_p .

Operations S_p , T_p , s_p are defined by the formulas

$$\begin{aligned} S_p\{a_k\} &\stackrel{\text{df}}{=} \{a_{k+1} + \{p_k\} * \{a_k\}\}, \\ T_p\{a_k\} &\stackrel{\text{df}}{=} \{T(\{a_k\} * \{E_1^{T(p_k)}\}) * \{E_1^{-T(p_k)}\}\}, \\ s_p\{a_k\} &\stackrel{\text{df}}{=} \{s\{a_k\}\} * \{E_1^{-T(p_k)}\}, \quad \{a_k\}, \{p_k\} \in C(N), \end{aligned}$$

where

$$T\{u_k\} \stackrel{\text{df}}{=} \begin{cases} 0 & \text{for } k=0 \\ u_{k-1} & \text{for } k=1, 2, \dots, \end{cases} \quad \{u_k\} \in C(N)$$

and

$$\{E_1^{T(p_k)}\} \stackrel{\text{df}}{=} \{v_k\}.$$

The sequence $\{v_k\}$ is defined by the recurrent formula

$$\begin{cases} v_0 = 1 \\ v_{k+1} = \sum_{i=0}^k \binom{k}{i} v_i p_{k-i}, \quad k \geq 1. \end{cases}$$

The multiplication $\circ$ has the form

$$\{x_k\} \circ \{y_k\} \stackrel{\text{df}}{=} \{E^{T(p_k)}\} * \{x_k\} * \{y_k\}.$$

Series $\sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$ convergent in interval $|x| < R$ and sequences $\{a_k\}$ of real numbers are isomorphic. The derivative d/dt is equivalent to operation P (see [2]).

Cauchy's multiplication of the series is evidently equivalent to the multiplication $*$ of the sequences.

If $\{p_k\} = (-1, 0, 0, \dots)$ then $S_p = \Delta$ then for the operation Δ there exists a multiplication defined by the formula (16), where the sequence $\{E_1^{T(-1, 0, 0, \dots)}\} \stackrel{\text{df}}{=} \{v_k\}$ is defined by the formula

$$\{v_k\} = \{(-1)^k\}.$$

From the examples presented it follows how many types of equations can be solved: ordinary differential equations, partial differential equations difference equations.

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