On additive arithmetical functions with values in the circle group

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1. We shall use the following standard notations: $\mathbf{N}$=natural numbers; Z=rational integers; Q_x=multiplicative group of positive rationals; $\mathbf{R}_x$=multiplicative group of positive reals; Q=additive group of rationals; $\mathbf{R}$=additive group of reals; T=one-dimensional circle group (torus), each of them in the usual topology.

Let G be an Abelian group, metrically compact in addition. A function $\varphi: \mathbf{N}\to G$ is called an additive numbertheoretical function if $\varphi(nm)=\varphi(n)+\varphi(m)$ holds for every coprime pairs of m and n, $m, n\in\mathbf{N}$. We shall say that φ is completely additive if $\varphi(mn)=\varphi(m)+\varphi(n)$ for any $m, n\in\mathbf{N}$.

Let x_ν $(\nu=1, 2, \ldots)$ be an infinite sequence in G. We shall say that it is of property D, if for any convergent subsequence x_{ν_l} $(l=1, 2, \ldots)$ the shifted subsequence x_{ν_l+1} $(l=1, 2, \ldots)$ is convergent too. We shall say that x_ν is of property K if $x_{\nu+1}-x_\nu\to 0$ $(\nu\to\infty)$. It is obvious that the sequences having property K all have property D and that the second set is much wider in general than the first one. In our paper [1] we proved that if $x_\nu=\varphi(\nu)$, φ is completely additive, then the fulfilment of D implies the fulfilment of K. This, by a recent result due to E. Wirsing allowed us to give a complete characterization of the completely additive functions having property D.

Our purpose in this paper is to solve this problem for the additive functions if $G=T$.

2. Theorem 1. *Let* $\varphi: \mathbf{N}\to T$ *be an additive function satisfying property D, i.e. the existence of* $\lim \varphi(n_k)=g$ $(n_1<n_2<\ldots)$ *implies the existence of* $\lim \varphi(n_k+1)=g'$. *Then*

$$\varphi(n) = u(n)+\tau\log n \pmod 1 \tag{2.1}$$

with a suitable $\tau\in\mathbf{R}$, *u is an additive function,* $u(m)=0$ *for every odd m,* $u(2^\alpha)=u(2)$ *for every* $\alpha\geqq 1$.

Furthermore, if $\tau\neq 0$, *then* $u(2^\alpha)=0$ *or* $1/2$.

PROOF. Let $\varphi\in D$ be fixed in the sequel. Let S denote the set of accumulation points of the sequence $a_n:=\varphi(n)$ $(n\in\mathbf{N})$, i.e. let $g\in S$ if there exists a subsequence $n_1<n_2<\ldots$ of natural numbers such that $g=\lim_k \varphi(n_k)=g$, then let

$$F(g) := g' = \lim_k \varphi(n_k+1).$$

Lemma 1. *The correspondence* F: $S\to S$ *is a function and* $F(S)=S$.

The proof of it was given in [1].

Let $p(n)$ and $P(n)$ denote the smallest and the largest prime factor of $n\in\mathbf{N}$. We shall say that a sequence $M:=\{M_1<M_2<\ldots\}$ $(M_\nu\in\mathbf{N})$ belongs to $\mathscr{M}_0$ if for every $d\in\mathbf{N}$ $M_\nu-1\equiv 0\,(\mathrm{mod}\,d)$ whenever ν is sufficiently large, $\nu>\nu_0(d, M)$. We shall say that a sequence $L:=\{L_1<L_2<\ldots\}$ belongs to $\mathscr{M}_k$ $(k\in Z)$ if $M(=L_{-k})=\{L_1-k<L_2-k<\ldots\}$ belongs to $\mathscr{M}_0$. Let $L:=\{L_1<L_2<\ldots\}\in\mathscr{M}_k$. We shall say that $L\in\tilde{M}_k$ $(\subseteq\mathscr{M}_k)$ if there exists $\lim_\nu \varphi(L_\nu):=:a(L)$.

Let H_k denote the set of the limit points $a(L)$ while L runs over $\tilde{M}_k$.

First we consider H_0. Let $M\in\tilde{M}_0$, $R=\{R_1<R_2<\ldots\}\in\tilde{M}_0$. For a given ν let j_ν be so large that $p(R_{j_\nu})>P(M_\nu)$, $j_\nu>j_{\nu-1}$; $j_0=0$. Then $(R_{j_\nu}, M_\nu)=1$, consequently $\varphi(R_{j_\nu}M_\nu)=\varphi(R_{j_\nu})+\varphi(M_\nu)$, and the sequence $\{S_\nu=R_{j_\nu}M_\nu\ (\nu=1, 2, \ldots)\}$ belongs to $\tilde{M}_0$. Hence it follows immediately that $\lim\varphi(M_\nu)=g_1$, $\lim\varphi(R_\nu)=g_2$ implies $\lim\varphi(S)=g_1+g_2\in H_0$. Then H_0 is a semigroup in T. It is clear that H_0 is closed. The closedness of H_0 implies that H_0 is a compact semigroup in T, and so by [4] (9.16) it must be a group. Consequently H_0 is a discrete group or $H_0=T$.

3. Continuation of the proof, case $H_0=T$

From Lemma 1 we get that $H_0=H_1=H_2=\ldots=T$.

Lemma 2. *Let* $R:=\{\varphi(n)|n\in\mathbf{N}\}$. *Then* $R+T\subseteq T$. *R is everywhere dense in T.*

Let $g\in T(=H_0)$, and $M\in\tilde{M}_0$ be such a sequence for which $a(M)=g$. Let $N\in\mathbf{N}$ be fixed. Then $(M_\nu, N)=1$ for every large ν, so $\varphi(M_\nu N)=\varphi(M_\nu)+\varphi(N)\to g+\varphi(N)$, i.e. $R+T\subseteq T$. Since $0\in T$, therefore $R\subseteq T$. It is obvious that R is everywhere dense in T.

Lemma 3. *F is continuous on T.*

For the proof see Lemma 5 in [1].

Let $a, b\in\mathbf{N}$.

$$l(a, b) := \varphi(ab)-\varphi(a)-\varphi(b). \tag{3.1}$$

If the common prime factors of a and b are $p_1, p_2, \ldots, p_r$, $a=p_1^{\alpha_1}\ldots p_r^{\alpha_r}a_1$, $b=p_1^{\beta_1}\ldots p_r^{\beta_1}b_1$, $(a_1, b_1)=1$, p_j+a_1, p_j+b_1 $(j=1, \ldots, r)$, then

$$l(a, b)=\sum_{j=1}^{r} l(p_j^{\alpha_j}, p_j^{\beta_j}) \tag{3.2}$$

Let now a, b, c, d, f be nonnegative integers such that $a+b=c+d$, $ab=cd+f$. Then

$$(x+a)(x+b)=(x+c)(x+d)+f \tag{3.3}$$

is an identity.

Let $g\in T$, $n_1<n_2<\ldots$ be a sequence in $\mathscr{M}_k$ $(k\geqq 0)$ such that $\varphi(n_\nu)\to g$.

Since $n_\nu-(k+1)\equiv 0 \pmod d$ if $\nu>\nu_0(d)$ for every d, and $(n_\nu+a, n_\nu+b)|b-a$, therefore $l(n_\nu+a, n_\nu+b)=l(a+k+1, b+k+1)$ for every large ν, consequently

$$\varphi\big((n_\nu+a)(n_\nu+b)\big) \to l(a+k+1, b+k+1)+\varphi(n_\nu+a)+\varphi(n_\nu+b)$$

for every large ν. Arguing similarly with c, d instead of a, b, we get that

(3.4)

$$l(a+k+1, b+k+1)+F^a(g)+F^b(g) = F^f[l(c+k+1, d+k+1)+F^c(g)+F^d(g)].$$

Hence we get that $l(c+k_1+1, d+k_2+1)=l(c+k_2+1, d+k_2+1)$ implies $l(a+k_1+1, b+k_1+1)=l(a+k_2+1, b+k_2+1)$.

Let now $q>2$ be a prime, $(a, q)=1$, $b-a=q$, $d=0$, $c=a+b=2a+q$. Then $(2a+q+1+k, 1+k)=1$ for at least one arithmetical progression $k \pmod{(2a+q)}$. Let us fix such an arithmetical progression $k\equiv k_0 \pmod{(2a+q)}$. Then $l(c+1+k, d+1+k)=0$ as $k\equiv k_0$. Let us consider now $l(a+1+k, b+1+k)$ while $k\equiv k_0 \pmod{(2a+q)}$. Since $b-a=q$, $(a+1+k, b+1+k)|b-a=q$, therefore $(a+1+k, b+1+k)=1$ or q. Since $(2a+q, q)=1$, therefore the system $a+1+k\neq 0 \pmod q$, $k\equiv k_0 \pmod{(2a+q)}$ is solvable. Hence we get that $l(a+1+k, b+1+k)=0$ as $k\equiv k_0 \pmod{(2a+q)}$. Similarly, the congruences $a+1+k\equiv 0 \pmod{q^\alpha}$, $a+1+k\not\equiv \not\equiv 0 \pmod{q^{\alpha+1}}$ $b+1+k\not\equiv 0 \bmod q^2$), $k\equiv k_0 \pmod c$ are solvable for every $\alpha\geqq 1$, we get that

$$l(q^\alpha, q) = 0 \quad (\alpha = 1, 2, \ldots).$$

This implies immediately that $\varphi(q^\alpha)=\alpha\varphi(q)$. So we have proved the following

Lemma 4. *$\varphi(mn)=\varphi(m)+\varphi(n)$ for every pairs of odd integers m, n.*

Lemma 5. *We have*

$$F[g]+F^2[g] = F^2\big[g+F^3[g]\big]. \tag{3.5}$$

Let n_ν be so chosen that $\varphi(n_\nu)\to g$. Starting from the identity $(x+1)(x+2)= =x(x+3)+2$, taking into account Lemma 4, we get that

$$\varphi\big((n_\nu+1)(n_\nu+2)\big) \to F(g)+F^2(g),$$

$$\varphi\big(n_\nu(n_\nu+3)\big) \to g+F^3[g], \quad \varphi\big(n_\nu(n_\nu+3)+2\big) \to F^2\big[g+F^3[g]\big].$$

Let now η be such an element in T for which $F^3(\eta)=0$. The existence of such an η is almost obvious (see [1]). Then, if we put $g=\eta$ into (3.5), we get $F(\eta)=0$, which implies

Lemma 6.

$$F^2[0] = 0. \tag{3.6}$$

Let now k be an odd integer, $M=\{M_1<M_2<\ldots\}\in\tilde{\mathcal{M}}_0$ such that $\varphi(M_\nu)\to \to -\varphi(k)$. Then

$$\varphi(kM_\nu) = \varphi(k)+\varphi(M_\nu) \to 0,$$

$$\varphi(kM_\nu+k) \to F^k[0] = F[0],$$

$$\varphi(kM_\nu+k) = \varphi(k)+\varphi(M_\nu+1) \to \varphi(k)+F[-\varphi(k)].$$

This implies that

$$F[-\varphi(k)] = -\varphi(k) + F[0]. \tag{3.7}$$

In [1] we proved that F is continuous. Since $H_0 = T$, therefore $\varphi(k)$ $(k=1, 3, 5, \ldots)$ is everywhere dense in T. Since T is a group therefore $\{-\varphi(k)\}$ is everywhere dense as well. Consequently we have

Lemma 7.

$$F(g) = g + F[0]. \tag{3.8}$$

Hence it follows immediately that

$$F^2[g] = F\big[F(g)+F[0]\big] = F^2[g]+2F[0],$$

i.e. that

$$2F[0] = 0. \tag{3.9}$$

So $F[0]=\gamma$, $\gamma=0$ or $1/2$.

Lemma 8. *We have*

$$\lim_{n\to\infty} \{\varphi(n+1)-\varphi(n)\} = \gamma$$

Assuming the contrary, there exists a subsequence $n_1<n_2<\ldots$ of positive integers such that $\Delta\varphi(n_j)=\varphi(n_j+1)-\varphi(n_j)\to h$, $h\neq\gamma$. We can choose a suitable convergent subsequence of $\varphi(n_{j_l})$, $\varphi(n_{j_l})\to g$. Then $\varphi(n_{j_l}+1)\to g+h$, $F[g]=g+h$, $h=F[0]$. This contradicts to $F[0]=\gamma$.

Lemma 9. *If* $\psi\colon \mathbf{N}\to T$ *is additive, and* $\Delta\psi(n):=\psi(n+1)-\psi(n)\to 0$ $(n\to\infty)$, *then* ψ *is completely additve.*

For the proof see e.g. [5].

We shall need the following theorem due to E. WIRSING [3] which we quote now as

Lemma 10. *If* $\psi\colon \mathbf{N}\to T$ *is completely additive and* $\Delta\psi(n)\to 0$, *then* $\psi(n)\equiv$ $\equiv\tau\log n \pmod 1$ *with a suitable* $\tau\in R$.

Now we finish the proof of the theorem. If $\gamma=0$, then $F[0]=0$, Lemma 9 and 10 implies that (2.1) holds with $u(n)=0$ $(n=1, 2, \ldots)$. Let us assume that $\gamma=1/2$. Then $\psi(n):=2\varphi(n)$ satisfies the conditions of Lemma 10, consequently $\psi(n)=$ $=K\log n \pmod 1$. Consequently $\varphi(n)=\tau\log n+u(n) \pmod 1$, $\tau=\dfrac{K}{2}$, where $2u(n)=0$ for every n. Since $\Delta\varphi(n)=o(1)+\Delta u(n)$, $\Delta\varphi(n)\to 1/2$, therefore $\Delta u(n)\to 1/2$, which by $u(n)=0$ or $1/2$ gives that $u(n+2)-u(n)=0$ for every large n. Then $u(n)$ is ultimately periodic mod 2, completely additive on the set of odd numbers, conse-quently $u(n^2)=u(n)$ $(n, 2)=1$, i.e. $u(n)=0$ if $(n, 2)=1$. Furthermore $u(n+1)-u(n)=1/2$ for every large n, consequently $u(2)=u(2^2)=\ldots=1/2$.

The proof is finished in the case $H_0=T$.

4. Continuation of the proof, case $H_0 \neq T$

In this case H_0 is a discrete subgroup, the case $H_0=\{0\}$ may be occur.

Let A be the set of those sequences $x=\{x_1, x_2, \ldots\}$, $x_\nu \in \mathbf{N}$ for which there exists $a(x)=\lim_\nu \varphi(x_\nu)$, and $p(x_\nu)\to\infty$ as $\nu\to\infty$. Let A be the set of limit points $a(x)$, $x\in A$. Since $\tilde{\mathcal{M}}_0 \subseteq A$, $\tilde{\mathcal{M}}_{-2}\subseteq A$, therefore $H_{-2}\subseteq A$, $H_0\subseteq A$. It is obvious that A is a closed semigroup, and so it is a group. To prove this we need only to repeat the argument has been done in section 2. Let now $x\in A$, $R=\{R_1<R_2<\ldots\}\in \in\tilde{M}_{-1}$. Let j_ν be so large that $j_\nu>j_{\nu-1}$, $P(R)<p(x_{j_\nu})$. Then the sequence $S_\nu=R_\nu x_{j_\nu}$ belongs to $\tilde{\mathcal{M}}_{-1}$, $(R_\nu, x_{j_\nu})=1$, and so $a(R)+a(x)\in H_{-1}$. So we have

$$A+H_{-1}\subseteqq H_{-1}, \tag{4.1}$$

and from $H_{-2}\subseteq A$, $H_0\subseteq A$, we get that

$$H_{-2}+H_{-1}\subseteqq H_{-1},\quad H_0+H_{-1}\subseteqq H_{-1}. \tag{4.2}$$

Let us consider now H_{-2}. Let $x, y\in\tilde{\mathcal{M}}_{-2}$, $a(x)=h_1$, $a(y)=h_2$. Then the sequence $z_\nu=x_\nu y_{j_\nu}$, $p(y_{j_\nu})>P(x_\nu)$ belongs to $\tilde{\mathcal{M}}_0$, and $a(z)=a(x)+a(y)$, i.e.

$$h_1+h_2\in H_0, \quad \text{whenever} \quad h_1, h_2\in H_{-2}. \tag{4.3}$$

Similarly, if we take $x\in\tilde{\mathcal{M}}_{-2}$, $y\in\tilde{\mathcal{M}}_0$, we get that $z\in\mathcal{M}_{-2}$, consequently

$$h+g\in H_{-2} \quad \text{whenever} \quad h\in H_{-2},\ g\in H_0. \tag{4.4}$$

Since F^2 is a function, $F^2(H_{-2})=H_0$, therefore card $(H_{-2})\geqq$ card (H_0). Let us fix h_2, and let h to run over H_{-2} in (4.3). Then we get immediately that

$$\text{card}\,(H_{-2}) = \text{card}\,(H_0). \tag{4.5}$$

But this implies that

$$\text{card}\,(H_{-2}) = \text{card}\,(H_{-1}) \tag{4.6}$$

From (4.1) we get that card $(A)\leqq$ card (H_{-1}). From (4.5), (4.6), and from $H_0\cup H_2\subseteq A$ we get that $H_{-2}=H_0$.

Let now $M=\{M_1<M_2<\ldots\}\in\tilde{\mathcal{M}}_0$, $a(M)=g$, $k>1$. Then $\varphi(M_\nu+k)\to F^k[g]$.

We may assume that $M_\nu-1\equiv 0 \pmod{(k+1)}$ for $\nu\geqq 1$. Then the sequence $N_\nu=\dfrac{M_\nu-1}{k+1}+1$ belongs to $\mathcal{M}_0$, consequently a rarified subsequence of it belongs to $\tilde{M}_0$, $\varphi(N_{j\nu})\to g^*\in H_0$. Since $\varphi(M_\nu+k)=\varphi\big((k+1)N_\nu\big)$, we get that $F^k[g]= =g^*+\varphi(k+1)$, and so $H_k=F^k[H_0]\subseteq H_0+\varphi(k+1)$. Observing that the function F^2: $H_{-2}\to H_0$ is one to one, we have that card $(H_k)=$card (H_0) for every k, consequently

$$H_k = H_0+\varphi(k+1). \tag{4.7}$$

Since $H_0=H_2=H_4=\ldots$, from (4.7) we get that $\varphi(2n+1)\in H_0$ whenever $n\in\mathbf{N}$. Furthermore, $H_1=H_3=\ldots$ that by (4.7) implies that $\varphi(2k)-\varphi(2l)\in H_0$ for each $k, l\in\mathbf{N}$.

Since $H_{-2}=H_0$, there exists a sequence $\{y_\nu\}\in\tilde{\mathcal{M}}_{-2}$ such that $\varphi(y_\nu)\to 0$. Let $z_\nu=y_\nu-3$, and z_{n_ν} be so rarified that $\lim\varphi(z_{n_j})=h$ there exists. Since $3+y_\nu$, therefore $(z_\nu, y_\nu)=1$ for each large ν. Starting from the identity $2+x(x+3)=(x+1)(x+2)$, substituting $x=z_{n_\nu}$, we deduce that

$$F^2\big[h+F^3[h]\big]=F[h]+F^2[h],$$

that by $0=F^3[h]=\lim\varphi(y_\nu)$ gives that $F(h)=0$, whence $F^2[0]=0$ immediately follows. Since in this case $R=\{\varphi(n)|n\in\mathbf{N}\}$ is a sicsrete set, for a sequence $n_1<n_2<\ldots$ $\ldots\varphi(n_\nu)$ is convergent if and only if $\varphi(n_\nu)=\varphi(n_{\nu+1})$ for every large ν. Let $\mathcal{B}$ denote the set of all $n\in\mathbf{N}$ for which $\varphi(n)=0$. $\mathcal{B}$ contains infinitely many element. From $F^2[0]=0$ it follows immediately that $n\in\mathcal{B}$, n is large enough implies $n+2\in\mathcal{B}$. Since $\mathcal{B}$ contains infinitely many odd elements, therefore $\varphi(n)=0$ for every large odd integer. From the additivity we get that $\varphi(n)=0$ for every odd n. Let now $n_\nu=2^\alpha\nu-1$ $(\nu=1, 3, \ldots)$.

Then $\varphi(n_\nu)=0$ $(\nu=1, 3, \ldots)$, consequently $\varphi(n_\nu+1)=\varphi(2^\alpha)=F[0]$ $(\nu=1, 2, \ldots)$, i.e. $\varphi(2)=\varphi(2^2)=\varphi(2^3)=\ldots$.

The theorem is proved.

References

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