

***N*-recurrent *V*-manifolds and Takeno's conjecture**

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1. Introduction

TAKENO and KITAMURA [1] have introduced the *V*-manifolds and they have developed the theory of *V*-space-times (cf. references in [2]). Later TAKENO [2] has studied the recurrency conditions for *V*-manifolds. He found the same conditions for 1, 2, and 3-recurrent *V*-manifolds, and he conjectures that this condition is also true for *n*-recurrent manifolds ($n \geq 4$). In this paper, the author will formulate again the recurrency conditions in a some different way, to find among other theorems a proof of Takeno's conjecture. It will be shown here that the same results as in [2] can be found easily without computing $\nabla^2 R$ and $\nabla^3 R$ in a local chart. The computations for finding the components of different tensors were realized with an IBM-370 using methods developed by the author [3].

A *V*-manifold is defined here to have, in a local coordinate system, the following non-vanishing components for the covariant metric tensor:

$$(1-1) \quad g_{11} = -1 \quad g_{22} = -B \quad g_{33} = -C \quad g_{44} = D$$

where *B*, *C* and *D* are functions of a single variable *x*, and are of class C^∞ :

2. Recurrent and symmetric manifolds

Symmetric manifolds, in short *s*-manifolds were introduced and completely investigated by CARTAN [4]. Recurrent manifolds, in short *r*-manifolds, were introduced by RUSE [5] and most of their properties investigated by WALKER [6].

Let *R* be the Riemann—Christoffel tensor defined by: $R(x, y)z = [\nabla_x, \nabla_y]z - \nabla_{[x, y]}z$ where *x*, *y*, *z* ∈ $\mathfrak{R}$ are vector fields and ∇ is the connexion on the manifold. It will be supposed here that the torsion: $T(x, y) = \nabla_x y - \nabla_y x - [x, y] \equiv 0$, is zero.

The recurrency condition can be formulated:

$$(2-1) \quad \nabla R = R \otimes a$$

where *a* is a 1-form.

In a similar way an r^n -manifold is defined, when the following condition is satisfied:

$$(2-2) \quad \nabla^{(n)} R = R \otimes t$$

where $t \in \otimes^n \mathfrak{R}^*$ is a tensor field of order *n*.

We will consider s -manifolds as special cases of r -manifolds; they arise if in (2-1) $\mathbf{a} \equiv 0$, i.e.: $\nabla R = 0$; and we get an s^n -manifold if $\mathbf{t} \equiv 0$ in (2-1), i.e.: $\nabla^{(n)} R = 0$.

The properties of r -manifolds have been investigated by several authors (see references in [7] and [8]). In this paper, we will use some of these properties which are well known. To simplify the presentation these will be given in form of propositions without proof.

Proposition 2-1. Any r -manifold is a r^n -manifold i.e.: if $\nabla R = R \otimes \mathbf{a}$ then $\nabla^{(n)} R = R \otimes \mathbf{t}$ where $\mathbf{t} = f(\mathbf{a}, \nabla \mathbf{a}, \dots, \nabla^{(n-1)} \mathbf{a})$.

In particular for r^2 -manifolds the recurrence tensor $\mathbf{b}$ is given by $\mathbf{b} = \mathbf{a} \otimes \mathbf{a} + \nabla \mathbf{a}$.

Proposition 2-2. If $\nabla R = 0$ then $\nabla^{(n)} R = 0$.

Proposition 2-3. For r -manifolds the recurrence's 1-form is closed, i.e. $\mathbf{a} = d\theta$;

Proposition 2-4. If $R \neq 0$ (scalar curvature) then the 1-form of recurrence is given by $\mathbf{a} = d \ln R$.

Proposition 2-5. If $R \neq 0$ then any r^2 -manifold is necessarily an r -manifold.

Proposition 2-6. For non-simple r -manifolds the recurrence 1-form $\mathbf{a}$ is isotropic and recurrent. i.e.: $\nabla \mathbf{a} = \tau \cdot \mathbf{a} \otimes \mathbf{a}$, and $g(\mathbf{a}, \mathbf{a}) = 0$.

Proposition 2-7. For r^2 -manifolds the recurrence tensor is symmetric, $\mathbf{A}\mathbf{b} = 0$.

Let us now investigate the conditions for an r -manifold, with $\mathbf{a} \neq 0$, to be an s^2 -manifold.

We must have $\mathbf{b} = 0 \Leftrightarrow \nabla \mathbf{a} = \tau \cdot \mathbf{a} \otimes \mathbf{a}$ and we can state:

Proposition 2-8. The only r -manifolds which can be s^2 -manifolds are the non-simple ones in Walker's classification [6].

The properties of r -vector fields can be found elsewhere [9].

Proposition 2-9. A manifold with constant curvature is necessarily an s -manifold.

Proposition 2-10. In an r -manifold there exists a tensor $S \in \mathfrak{T}$ which has the same properties of symmetry as R and such that it is parallel $\nabla S = 0$ and is collinear to R ; i.e.: $S = \varphi R$ and φ is given by: $\varphi = \exp \{k\theta\}$ where $d\theta = \mathbf{a}$, and if in particular $R \neq 0$ then $\varphi = \tau \cdot R$.

PROOF. Since $\nabla S = \nabla \varphi \otimes R + \varphi \cdot \nabla R$ and $\nabla R = R \otimes \mathbf{a}$, we have:

$$(2-3) \quad \nabla \varphi + \varphi \cdot \mathbf{a} = 0 \Rightarrow \mathbf{a} = d \ln \varphi$$

Now as $\mathbf{a} = d\theta$ we get: $\varphi = e^{k\theta}$, and we have the first part of proposition 10.

The second part follows immediately from proposition 4. Because $\mathbf{a} = d \ln R$, from (2-3) we get $d \ln \varphi = d \ln R$.

3. Recurrency conditions for V -manifolds

In a coordinate system the recurrency condition can be written, as follows:

$$(3-1) \quad \nabla_{\mu} R_{\alpha\beta\varrho\sigma} = a_{\mu} R_{\alpha\beta\varrho\sigma}$$

Proposition 3-1. *For indices $\alpha, \beta, \varrho, \sigma$ and μ fixed, we have:*

- i) *if $R_{\alpha\beta\varrho\sigma} = 0$ then $\nabla_{\mu} R_{\alpha\beta\varrho\sigma} = 0$*
- ii) *if $\nabla_{\mu} R_{\alpha\beta\varrho\sigma} = 0$ and $R_{\alpha\beta\varrho\sigma} \neq 0$ then $a_{\mu} = 0$.*

Let $X(x)$ be an arbitrary non-vanishing function of x . We put:

$$(3-2) \quad \varphi_n(X) \equiv X^{(n+1)}/X^{(n)}$$

$$\text{for } n \in \mathbb{Z}^+ \text{ and } X^{(n)} \equiv \frac{d^n}{dx^n} X(x), \quad X^{(0)} \equiv X(x).$$

By straightforward computation it will be found that:

$$(3-3) \quad \frac{d}{dx} \varphi_n = \varphi_n(\varphi_{n+1} - \varphi_n) \quad \therefore \quad n \in \mathbb{Z}^+$$

We define the following functionals:

$$(3-4) \quad \begin{aligned} F(X) &\equiv \frac{1}{4} \varphi_0(X) [\varphi_0(X) - 2\varphi_1(X)] \\ H(X) &\equiv \frac{1}{2} \varphi_0(X) [\varphi_0^2(X) - 2\varphi_1(X) \varphi_0(X) + \varphi_1(X) \varphi_2(X)] \end{aligned}$$

$$G(X, Y) \equiv \frac{1}{4} \varphi_0(X) \cdot \varphi_0(Y)$$

$$J(X, Y) \equiv \frac{1}{2} [\varphi_0(X) + \varphi_0(Y) - 2\varphi_1(X)].$$

We can now write the non-vanishing independents components of R :

$$(3-5) \quad \begin{aligned} R_{1212} &= -B \cdot F(B) \\ R_{1414} &= D \cdot F(D) \\ R_{2323} &= B \cdot C \cdot G(B, C) \\ R_{2424} &= -B \cdot D \cdot G(B, D) \\ R_{3131} &= -C \cdot F(C) \\ R_{3434} &= C \cdot D \cdot G(C, D) \end{aligned}$$

For the components $\nabla_1 R$ we have:

$$\begin{aligned}
 R_{1212/1} &= B \cdot H(B) \\
 R_{1414/1} &= -D \cdot H(D) \\
 R_{2323/1} &= -B \cdot C \cdot G(B, C) [J(B, C) + J(C, B)] \\
 R_{2424/1} &= B \cdot D \cdot G(B, D) [J(B, D) + J(D, B)] \\
 R_{3131/1} &= C \cdot H(C) \\
 R_{3434/1} &= D \cdot C \cdot G(C, D) \cdot [J(C, D) + J(D, C)]
 \end{aligned}
 \tag{3-6}$$

and for the other components:

$$\begin{aligned}
 R_{1223/3} &= B \cdot C \cdot G(B, C) \cdot J(B, C) \\
 R_{1224/4} &= -B \cdot D \cdot G(B, D) \cdot J(B, D) \\
 R_{1424/2} &= B \cdot D \cdot G(B, D) \cdot J(D, B) \\
 R_{1434/3} &= D \cdot C \cdot G(C, D) \cdot J(D, C) \\
 R_{3123/2} &= C \cdot B \cdot G(C, B) \cdot J(C, B) \\
 R_{3134/4} &= C \cdot D \cdot G(D, C) \cdot J(C, D)
 \end{aligned}
 \tag{3-7}$$

and the scalar curvature is given by:

$$R = -2[F(B) + F(C) + F(D) - G(B, C) - G(B, D) - G(D, C)]
 \tag{3-8}$$

By inspection of (3-5), (3-6) and (3-7), we get for example:

$$R_{1212} \neq 0 \quad \text{and} \quad R_{1212/\alpha} = 0 \quad \text{for} \quad \alpha = 2, 3, 4$$

and from proposition (3-1) ii) we get:

$$a_2 = a_3 = a_4 = 0.$$

Now we can state

Proposition 3-2. *For V-manifolds the most general form for the recurrency's 1-form is given in a local chart by: $a_\mu = \delta_\mu^1 a_1$.*

Let us consider now the different cases of V-manifolds when all of the functions B, C and D are not constants.

All of the $\bar{B}, \bar{C}$ and $\bar{D}$ (here $X \equiv \frac{dX(x)}{dx}$) cannot vanish, otherwise the manifold is flat.

Type I: All functions different from zero $\bar{B} \cdot \bar{C} \cdot \bar{D} \neq 0$.

Type II: At most one vanishing function. Subcases:

$$II_a: \bar{B} = 0, \quad \bar{C} \cdot \bar{D} \neq 0$$

$$II_b: \bar{C} = 0, \quad \bar{B} \cdot \bar{D} \neq 0$$

$$II_c: \bar{D} = 0, \quad \bar{B} \cdot \bar{C} \neq 0$$

Type III: At most one non-vanishing function Subcases:

$$III_a: \bar{B} \neq 0, \quad \bar{C} = \bar{D} = 0$$

$$III_b: \bar{C} \neq 0, \quad \bar{B} = \bar{D} = 0$$

$$III_c: \bar{D} \neq 0, \quad \bar{B} = \bar{C} = 0$$

These three different types will be investigated in the following three sections.

4. Recurrency conditions and solutions for type I

We have $\bar{B} \cdot \bar{C} \cdot \bar{D} \neq 0$ and from (3-1), (3-5) and (3-6) we get:

$$A1) \quad a_1 = -\frac{H(X)}{F(X)} = J(X, Y) + J(Y, X)$$

for X any of B, C , or D , and $X \neq Y$. From (3-7) and proposition (3-1) i) we get:

$$B1) \quad J(X, Y) = 0$$

for X, Y any B, C , or D , and $X \neq Y$. When B1 is satisfy follows that

$$i) \quad a_1 = 0$$

$$ii) \quad H(X) = 0, \quad \text{for } X \text{ any } B, C \text{ or } D.$$

Let us investigated the case when condition B1 is satisfied. It follows from the definition of J in (3-4) that:

$$(4-2) \quad J(X, Y) = 0 \Leftrightarrow \varphi_0(X) + \varphi_0(Y) = 2\varphi_1(X)$$

and since (4-2) must be true when (X, Y) is replaced by any of the couples (B, C) , (B, D) , (C, B) , (C, D) , (D, B) or (D, C) , we must have

$$(4-3) \quad \varphi_0(X) = \varphi_0(Y)$$

for any X and Y . Then the equation (4-2) become

$$(4-4) \quad \varphi_0(X) = \varphi_1(X)$$

and with the help of definition (3-2) the solution of this differential functional equation, can be found easily to be:

$$(4-5) \quad X(x) = x \cdot e^{kx}$$

Now condition ii) of (4-1) imposes more restrictions on (4-5).

$$(4-6) \quad \text{If } H(X) = 0 \quad \text{then} \quad \varphi_0^2(X) - 2\varphi_0(X)\varphi_1(X) + \varphi_1(X) = 0$$

and with (4-4), (4-6) become

$$(4-7) \quad \varphi_0(X) = \varphi_2(X)$$

From (3-2) we have $\varphi_n(X) = k$ and then from (3-4) and conditions (4-4) and (4-7) we get: $F(X) = -G(X, Y) = -k^2/4$.

The scalar curvature can be written:

$$(4-8) \quad R = -\varphi_0(B) \cdot J(B, C) - \varphi_0(C) \cdot J(C, D) - \varphi_0(D) \cdot J(D, B)$$

from B1 we get $R=0$.

We can formulate these results in the four following propositions.

Proposition 4-1. *Recurrent V-manifolds of type I impose the following restrictions on the functions:*

$$B = x_1 e^{kx}, \quad C = x_2 e^{kx} \quad \text{and} \quad D = x_3 e^{kx}$$

Proposition 4-2. *Recurrent V-manifolds of type I are conformal to flat manifold. i.e.: $W=0$ (Weyl's tensor).*

Proposition 4-3. *Recurrent V-manifolds of type I are manifolds of constant curvature. i.e. they verifying*

$$R_{\alpha\beta\varrho\sigma} = k(g_{\alpha\varrho}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\varrho}).$$

Proposition 4-4. *Recurrent V-manifolds of type I are s-manifolds i.e. the recurrency's 1-form is $a \neq 0$.*

The last proposition follows also from proposition (4-3) and (2.9).

5. *Recurrency conditions for type II.* The solutions are similar for each one of the different subcases. For each subcase we have only one function constant.

From (3-1), (3-5) and (3-6), we get:

$$A2) \quad a_1 = -\frac{H(X)}{F(X)} = -\frac{H(Y)}{F(Y)} = J(X, Y) + J(Y, X)$$

and from (3-7) and proposition 3-1:

$$B2) \quad J(X, Y) = J(Y, X) = 0$$

here X and Y are different and they replace for each case Π_a , Π_b , Π_c , and (C, D) , (B, D) , (B, C) respectively.

When B2 is satisfied we get:

$$i) \quad a_1 = 0$$

(5-1)

$$ii) \quad H(X) = H(Y) = 0.$$

Using definition (3-2) and with the help of (3-3) it is a straightforward matter to verify that if

$$(5-2) \quad \frac{d}{dx} F = -H$$

then condition (5-1) ii) is equivalent to:

$$(5-3) \quad F = C^{te} \neq 0.$$

Otherwise $X=(x+b)^2$, $Y=C^{te}$ i.e. the manifold is flat $R=0$.

The integral of the differential equation (5-3) can easily be found if we note that $\varphi_0^1 = \varphi_0 \cdot \varphi_1 - \varphi_0^2$, and if we replace by (5-3) and put the constant $-\frac{1}{4}k^2$ we get the differential equation:

$$\frac{d\varphi_0}{k^2 - \varphi_0^2} = \frac{1}{2} dx.$$

The solutions are

- (5-4) i) $\varphi_0^2 < k^2, \quad k^2 > 0, \quad X(x) = \cos h^2 z$
 ii) $\varphi_0^2 < k^2, \quad k^2 < 0 \Rightarrow X(x) = \cos^2 z$
 iii) $\varphi_0^2 > k^2, \quad k^2 > 0 \Rightarrow X(x) = \sin h^2 z$
 iv) $\varphi_0^2 > k^2, \quad k^2 < 0 \Rightarrow X(x) = \sin^2 z$
 v) $k^2 = 0 \Rightarrow X(x) = (x+b)^2$

where $z = \frac{1}{2} kx + c$, k and c are constants.

If we replace the solution $X(x)$ in the differential equation $J(X, Y) = 0$ we get: $\varphi_0(Y) = 2\varphi_1(X) - \varphi_0(X)$, and the solutions for the above five cases are respectively:

- i) $Y(x) = \sin h^2 z$
 ii) $Y(x) = \sin^2 z$
 iii) $Y(x) = \cos h^2 z$
 iv) $Y(x) = \cos^2 z$
 v) $Y = C^{te}$.

In each one of the four cases, the conditions B2 and (5-1) are identically satisfied.

Proposition 5-1. *Recurrent V-manifold of type II impose the followin restriction on the functions:*

- a) $B = X, C = Y$ and $D = C^{te}$
 b) $B = Y, C = X$ and $D = C^{te}$

with $X = \cos h^2(k_1 x + k_2)$ and $Y = \sin h^2(k_1 x + k_2)$ where k_1 is a complex number with either the real or imaginary part vanishing.

Proposition 5-2. *Recurrent V-manifolds of type II are s-manifolds.*

6. Recurrency conditions for type III.

In these three cases we have only one non-constant component for the covariant metric tensor. We get only one condition, namely:

$$A3) \quad a_1 = -\frac{H(X)}{F(X)}$$

All other components of R and ∇R are identically vanishing.

With (5-3) condition A3 can be written:

$$a_1 = \varphi_0(F)$$

By proposition 2-4 we have the equivalent condition:

$$a_1 = \partial_1 \ln R$$

where

$$\begin{aligned} R &= -\frac{1}{2} \varphi_0(\varphi_0 - 2\varphi_1) \\ &= -2F \end{aligned}$$

Only when $F = C^{te} \neq 0$ is the manifold an s -manifold. This differential equation was solutioned in section 5, and the solution is given by (5-4).

When $F = 0$ the manifold is flat; then $X(x) = (x+b)^2$.

Proposition 6-1. *V-manifolds of type III are always r -manifolds. In particular we have:*

Proposition 6-2. *V-manifolds of III with $F = C^{te}$ are necessarily s -manifolds and the restrictions are: one of B, C and D is equal to X and the other are constants. Here $X(x) = \cos h^2 z$ or $X(x) = \sin h^2 z$.*

Proposition 6-3. *When $F \neq C^{te}$ then the scalar curvature is non-constant and the manifold is properly ($a \neq 0$) an r -manifold. The recurrency 1-form is given in the local chart by:*

$$a_x = \delta_x^1 \partial_1 R.$$

7. Takeno's conjecture

The results in the last three sections can be summarized by the table:

Type	Conditions on B, C and D		Solution	Recurrency 1-form	Manifold
I	all non-constants		$X = xe^{kx}$	$a = 0$	s -manifold
II	only one constant		$X = \cos h^2(kx + c)$ $Y = \sin h^2(kx + c)$	$a = 0$	s -manifold
III	s	only one	$F = C^{te} \neq 0$	$a = 0$	s -manifold
	r	non-constant	$F \neq C^{te}$	any other that $X \neq C^{te}$, $\cos h^2 z$, $\sin h^2 z$ or e^{kx}	$a_x = \delta_x^1 \partial_1 R$ r -manifold

For type I, II and III_s we can apply propositions 2-2. They are always s^n -manifolds.

For type III_r , we can apply propositions 2-1, 2-3, 2-4 and 2-5.

Let us investigate when a III_r -manifold is an s^2 -manifold. The square of the norm of the 1-form is $g(a, a) = a_1^2$ i.e.: it is always different from zero. Now we can apply propositions 2-6, 2-7, 2-8, 2-9 and 2-10.

We can state the following theorem:

Theorem 7-1. *Non-symmetric r^n -manifolds are necessarily of type III_r , i.e.: $F \neq C^{te}$, they are always non-symmetric r -manifolds.*

The following corollary asserts that Takeno's conjecture. cf. [2] is true.

Corollary 7-2. *For V -manifolds the conditions for r^n -manifolds are the same for any $n \geq 1$.*

In particular Takeno's theorems cf. [2], [5-1], [5-2], [7-3] and [7-4] are here obtained without computing $\nabla^2 R$ and $\nabla^3 R$ and they can be generalized for any $n \geq 1$.

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