

Contravariant integration, line integrals

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Abstract

In this paper an integral operator for tensor fields is introduced over the curves such that: 1° The result of integration is also a tensor field having one upper index more, 2° It is a generalization of the usual integral, and 3° The new operator acts over the differential 1-forms as a usual integration. The problem reduces to a system of Volterra's integral equations. Also some other properties of this integration are proved, and its geometrical interpretation is obtained.

1. Introduction

In this paper we shall solve the following problem. Let $\tau: [0, 1] \rightarrow M_n$ be an arbitrary smooth curve on a differentiable manifold endowed with an affine connection, and let a differentiable tensor field A of type (r, s) be given along the curve τ . The problem is to find an integral operator $\hat{\int}$ such that in local coordinate systems along the curve τ

(i) $\hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k$ should be components of a tensor field of type $(r+1, s)$.

(ii) In the special case when all components of the connection are identically zero, the problem should reduce to the usual integration, i.e.

$$(1.1) \quad \Gamma_{jk}^i \equiv 0 \Rightarrow \hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k = \int_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k$$

(iii) In the case of integration of an 1-form $A_i dx^i$ it should be

$$(1.2) \quad \hat{\int}_{\tau(0)}^{\tau(t)} A_i dx^i = \int_{\tau(0)}^{\tau(t)} A_i dx^i$$

In this paper a solution of this problem is found. The components

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k$$

are solutions of a system of integral equations, and they are unique. Apart from the above three properties we give some other properties and a geometric interpretation of this integral also is given. This integration is called contravariant, because the result of its action yields one upper index more.

For the sake of simplicity we do not write the boundaries of integration.

2. Solution in E^n , and its generalization

Let M_n be an n -dimensional euclidean space. Suppose that $(y^1, \dots, y^n) \equiv (y)$ is a rectangular Cartesian coordinate system and $(x^1, \dots, x^n) \equiv (x)$ is a curvilinear coordinate system. According to the system (y) we shall write components as A'_i, B'^j and for (x) : A_i, B^j , and so on.

From the transformation formula of the connection coefficients we obtain

$$(2.1) \quad \Gamma_{jk}^i(x) = \frac{\partial x^i}{\partial y^j} \frac{\partial^2 y^j}{\partial x^j \partial x^k},$$

for $\Gamma_{jk}^i(y) \equiv 0$. If we differentiate the identity

$$\delta_i^s = \frac{\partial y^s}{\partial x^u} \frac{\partial x^u}{\partial y^i}$$

by x^j , and use (2.1), we obtain the relation

$$(2.2) \quad \Gamma_{kj}^l = - \frac{\partial y^i}{\partial x^k} \frac{\partial y^r}{\partial x^j} \frac{\partial^2 x^l}{\partial y^i \partial y^r}.$$

We shall restrict ourselves to the case when A is a tensor field of type $(1, 1)$, in order to avoid long expressions. Partial integration of the equation

$$A_i^j = \frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} A'_\mu{}^\lambda$$

yields

$$(2.3) \quad \begin{aligned} \int_{\tau(0)}^{\tau(t)} A_i^j dx^k &= \int_{\tau(0)}^{\tau(t)} \frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} A'_\mu{}^\lambda \frac{\partial x^k}{\partial y^\nu} dy^\nu = \\ &= \frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \int_{\tau(0)}^{\tau(t)} A'_\mu{}^\lambda dy^\nu - \int_{\tau(0)}^{\tau(t)} \frac{\partial \left(\frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \right)}{\partial y^r} \left(\int_{\tau(0)}^{\tau(\theta)} A'_\mu{}^\lambda dy^\nu \right) dy^r. \end{aligned}$$

From conditions (i) and (ii) follows that

$$\int_{\tau(0)}^{\tau(t)} A'_\mu{}^\lambda dy^\nu = \frac{\partial x^p}{\partial y^\mu} \frac{\partial y^\lambda}{\partial x^q} \frac{\partial y^\nu}{\partial x^r} \int_{\tau(0)}^{\tau(t)} A_p^q dx^r$$

and

$$\frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \int_{\tau(0)}^{\tau(\theta)} A'_\mu{}^\lambda dy^\nu = \int_{\tau(0)}^{\tau(\theta)} A_i^j dx^k.$$

Putting this in (2.3), we obtain

$$\begin{aligned}
 \hat{\int}_{\tau(0)}^{\tau(t)} A_i^j dx^k &= \int_{\tau(0)}^{\tau(t)} A_i^j dx^k + \\
 &+ \int_{\tau(0)}^{\tau(t)} \frac{\partial \left(\frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \right)}{\partial y^t} \frac{\partial x^p}{\partial y^\mu} \frac{\partial y^\lambda}{\partial x^q} \frac{\partial y^\nu}{\partial x^r} \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q dx^r \right) \frac{\partial y^t}{\partial x^s} dx^s = \\
 &= \int_{\tau(0)}^{\tau(t)} A_i^j dx^k + \int_{\tau(0)}^{\tau(t)} \frac{\partial^2 x^k}{\partial y^\nu \partial y^t} \delta_q^j \frac{\partial y^\nu}{\partial x^r} \frac{\partial y^t}{\partial x^s} \delta_i^p \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q dx^r \right) dx^s + \\
 &+ \int_{\tau(0)}^{\tau(t)} \frac{\partial^2 x^j}{\partial y^\lambda \partial y^t} \delta_i^p \delta_r^k \frac{\partial y^t}{\partial x^s} \frac{\partial y^\lambda}{\partial x^q} \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q dx^r \right) dx^s + \\
 &+ \int_{\tau(0)}^{\tau(t)} \frac{\partial^2 y^\mu}{\partial x^i \partial x^u} \delta_q^j \delta_s^u \delta_r^k \frac{\partial x^p}{\partial y^\mu} \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q dx^r \right) dx^s.
 \end{aligned}$$

From (2.1) and (2.2) we obtain finally

$$\begin{aligned}
 (2.4) \quad \hat{\int}_{\tau(0)}^{\tau(t)} A_i^j dx^k &= \int_{\tau(0)}^{\tau(t)} A_i^j dx^k - \int_{\tau(0)}^{\tau(t)} \Gamma_{rs}^k \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_i^j dx^r \right) dx^s - \\
 &- \int_{\tau(0)}^{\tau(t)} \Gamma_{qs}^j \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_i^q dx^k \right) dx^s + \int_{\tau(0)}^{\tau(t)} \Gamma_{ir}^p \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^j dx^k \right) dx^r.
 \end{aligned}$$

In the general case by the same way we obtain the following system of equations

$$\begin{aligned}
 (2.5) \quad \hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k &= \int_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k - \int_{\tau(0)}^{\tau(t)} \Gamma_{l\lambda}^k \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^l \right) dx^\lambda - \\
 &- \int_{\tau(0)}^{\tau(t)} \Gamma_{l\lambda}^{i_1} \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1 \dots j_s}^{i_2 \dots i_r} dx^k \right) dx^\lambda - \dots - \int_{\tau(0)}^{\tau(t)} \Gamma_{l\lambda}^{i_r} \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1 \dots j_s}^{i_1 \dots i_{r-1}} dx^k \right) dx^\lambda + \\
 &+ \int_{\tau(0)}^{\tau(t)} \Gamma_{j_1\lambda}^l \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{i_1 \dots i_r j_2 \dots j_s}^{i_1 \dots i_r} dx^k \right) dx^\lambda + \dots + \int_{\tau(0)}^{\tau(t)} \Gamma_{j_s\lambda}^l \left(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{i_1 \dots i_r j_1 \dots j_{s-1}}^{i_1 \dots i_r} dx^k \right) dx^\lambda
 \end{aligned}$$

where $i_1, \dots, i_r, j_1, \dots, j_s, k \in \{1, 2, \dots, n\}$.

So the problem reduces to solve the system of integral equations (2.5) with unknown functions

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k.$$

It has a unique solution, because it is a Volterra system of integral equations and it can be solved by successive approximation ([1], [2], [3]). In fact (2.5) is a special case of Volterra system where the scores are functions only from the variable θ . So we see

that if in the euclidean space the problem raised in § 1 has solution, then it is a solution of the system (2.5), and this solution is unique.

Let us consider an M_n . Let U be a coordinate neighbourhood of it with local coordinates x^i . Consider (2.5) in U , where the tensor A and the curve τ are given, and the integral

$$\int_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_s} dx^k$$

is unknown. (2.5) has a unique solution for this integral.

Proposition. *This integral is a solution of our problem in U .*

PROOF. Condition (i) will be considered in § 3. Condition (ii) is obviously satisfied. From (2.5) we obtain

$$\int A_j dx^k = \int A_j dx^k - \int \Gamma_{il}^k \left(\int A_j dx^l \right) dx^i + \int \Gamma_{jl}^i \left(\int A_l dx^k \right) dx^i.$$

If we put here $j=k$ and sum up over k , we receive

$$\int A_k dx^k = \int A_k dx^k,$$

and the condition (iii) is satisfied. ■

We see that if V is a scalar, then

$$\int V dx^k = \int V dx^k - \int \Gamma_{il}^k \left(\int V dx^l \right) dx^i.$$

We see that in this case we do not integrate a scalar but a contravariant vector $V dx^k$. Integration of a scalar appears when we integrate the contraction $A_i dx^i$ and so in this case the contravariant integral is equal to the usual integral.

3. Tensor character of the contravariant integral

We prove that the solutions of the system (2.5) are tensor components, i.e. condition (i) holds. We shall restrict ourselves to the case when A is a covariant vector in order to avoid large expressions. In the general case when A is a tensor field of type (r, s) , the same result can be obtained in a similar way.

Let $x^1, \dots, x^n$ and $\bar{x}^1, \dots, \bar{x}^n$ be two coordinate systems. The components $\int A_i dx^k$ and $\int \bar{A}_i d\bar{x}^k$ are obtained as solutions of the following two systems

$$(3.1) \quad \int A_i dx^k = \int A_i dx^k - \int \Gamma_{il}^k \left(\int A_i dx^l \right) dx^i + \int \Gamma_{il}^r \left(\int A_r dx^k \right) dx^i$$

and

$$(3.2) \quad \int \bar{A}_i d\bar{x}^k = \int \bar{A}_i d\bar{x}^k - \int \bar{\Gamma}_{il}^k \left(\int \bar{A}_i d\bar{x}^l \right) d\bar{x}^i + \int \bar{\Gamma}_{il}^r \left(\int \bar{A}_r d\bar{x}^k \right) d\bar{x}^i$$

Our aim is to prove that

$$(3.3) \quad \int A_i dx^k = \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^s} \int \bar{A}_j d\bar{x}^s$$

for $i, k \in \{1, \dots, n\}$. It is sufficient to assume that (3.2) and (3.3) hold and to prove that the system (3.1) is satisfied, because the system (3.1) has a unique solution.

Putting

$$(3.4) \quad \bar{\Gamma}_{ij}^l = \frac{\partial \bar{x}^l}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^j} \Gamma_{st}^r + \frac{\partial \bar{x}^l}{\partial x^r} \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}$$

in (3.2) we get

$$\begin{aligned} \int \bar{A}_i d\bar{x}^k &= \int \bar{A}_i d\bar{x}^k - \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^r} \frac{\partial x^\nu}{\partial \bar{x}^l} \Gamma_{\mu\nu}^\lambda \left(\int \bar{A}_i d\bar{x}^r \right) d\bar{x}^l - \\ &- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \left(\int \bar{A}_i d\bar{x}^r \right) d\bar{x}^l + \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^l} \Gamma_{\mu\nu}^\lambda \left(\int \bar{A}_r d\bar{x}^k \right) d\bar{x}^l + \\ &+ \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^i \partial \bar{x}^l} \left(\int \bar{A}_r d\bar{x}^k \right) d\bar{x}^l. \end{aligned}$$

Expressing the integrals in the coordinate system (x)

$$\begin{aligned} \frac{\partial x^\mu}{\partial \bar{x}^l} \frac{\partial \bar{x}^k}{\partial x^\nu} \int A_u dx^\nu &= \int \frac{\partial x^s}{\partial \bar{x}^i} A_s \frac{\partial \bar{x}^k}{\partial x^r} dx^r - \\ &- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^r} \frac{\partial x^\nu}{\partial \bar{x}^l} \Gamma_{\mu\nu}^\lambda \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^r}{\partial x^\omega} \left(\int A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha - \\ &- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^r}{\partial x^\omega} \left(\int A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha + \\ &+ \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^r} \frac{\partial \bar{x}^k}{\partial x^\omega} \Gamma_{\mu\nu}^\lambda \left(\int A_s dx^\omega \right) dx^\nu + \\ &+ \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^i \partial \bar{x}^l} \frac{\partial x^s}{\partial \bar{x}^r} \frac{\partial \bar{x}^k}{\partial x^\omega} \left(\int A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha = \\ &= \int \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} A_s dx^\lambda - \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \Gamma_{\omega\nu}^\lambda \left(\int A_s dx^\omega \right) dx^\nu - \\ &- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^r}{\partial x^\omega} \frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \left(\int A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha + \\ &+ \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^i} \Gamma_{\mu\nu}^s \left(\int A_s dx^\lambda \right) dx^\nu + \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial \bar{x}^l}{\partial x^\alpha} \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^l} \left(\int A_s dx^\lambda \right) dx^\alpha. \end{aligned}$$

Putting here

$$\frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \frac{\partial \bar{x}^r}{\partial x^\omega} \frac{\partial \bar{x}^l}{\partial x^\alpha} = - \frac{\partial x^\lambda}{\partial \bar{x}^\mu} \frac{\partial^2 \bar{x}^\mu}{\partial x^\omega \partial x^\alpha}$$

we get

$$\begin{aligned} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} \int A_s dx^\lambda &= \int \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} A_s dx^\lambda - \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \Gamma_{\omega v}^\lambda \left(\int A_s dx^\omega \right) dx^v + \\ &+ \int \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^k}{\partial x^\lambda \partial x^v} \left(\int A_s dx^\lambda \right) dx^v + \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \Gamma_{sv}^\mu \left(\int A_\mu dx^\lambda \right) dx^v + \\ &+ \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial}{\partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \left(\int A_s dx^\lambda \right) dx^v. \end{aligned}$$

Using partial integration at several terms we obtain

$$\begin{aligned} (3.5) \quad \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} \int A_s dx^\lambda &= \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} \int A_s dx^\lambda - \int \frac{\partial^2 \bar{x}^k}{\partial x^\lambda \partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \left(\int A_s dx^\lambda \right) dx^v - \\ &- \int \left(\frac{\partial}{\partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \right) \frac{\partial \bar{x}^k}{\partial x^\lambda} \left(\int A_s dx^\lambda \right) dx^v - \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \int \Gamma_{\omega v}^\lambda \left(\int A_s dx^\omega \right) dx^v + \\ &+ \int \frac{\partial^2 \bar{x}^k}{\partial x^\lambda \partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \left(\int \Gamma_{\omega \alpha}^\lambda \left(\int A_s dx^\omega \right) dx^\alpha \right) dx^v + \\ &+ \int \left(\frac{\partial}{\partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \right) \frac{\partial \bar{x}^k}{\partial x^\lambda} \left(\int \Gamma_{\omega \alpha}^\lambda \left(\int A_s dx^\omega \right) dx^\alpha \right) dx^v + \\ &+ \int \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^k}{\partial x^\lambda \partial x^v} \left(\int A_s dx^\lambda \right) dx^v + \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \int \Gamma_{sv}^\mu \left(\int A_\mu dx^\lambda \right) dx^v - \\ &- \int \frac{\partial^2 \bar{x}^k}{\partial x^v \partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \left(\int \Gamma_{sz}^\mu \left(\int A_\mu dx^\lambda \right) dx^z \right) dx^v - \\ &- \int \frac{\partial}{\partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} \left(\int \Gamma_{sz}^\mu \left(\int A_\mu dx^\lambda \right) dx^z \right) dx^v + \\ &+ \int \left(\frac{\partial}{\partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \right) \frac{\partial \bar{x}^k}{\partial x^\lambda} \left(\int A_s dx^\lambda \right) dx^v. \end{aligned}$$

Let us denote

$$M_i^k = \int A_i dx^k - \int A_i dx^k + \int \Gamma_{il}^k \left(\int A_i dx^l \right) dx^l - \int \Gamma_{il}^k \left(\int A_l dx^k \right) dx^l.$$

Then (3.5) can be brought to the form

$$\begin{aligned} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} M_s^\lambda &= \int \frac{\partial^2 \bar{x}^k}{\partial x^\lambda \partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} M_s^\lambda dx^v + \int \left(\frac{\partial}{\partial x^v} \frac{\partial x^s}{\partial \bar{x}^i} \right) \frac{\partial \bar{x}^k}{\partial x^\lambda} M_s^\lambda dx^v, \\ \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} M_s^\lambda &= \int M_s^\lambda d \left(\frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \right). \end{aligned}$$

Differentiating this equality by the parameter t we obtain

$$\frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \frac{dM_s^\lambda}{dt} = 0.$$

So all the components M_s^λ must be constants along the curve $\tau(t)$. Since $M_s^\lambda = 0$ at the point $\tau(0)$, it follows that $M_s^\lambda = 0$ along the whole curve. So we have proved that the system (3.1) is satisfied.

4. Some properties of the contravariant integration

Since the system (2.5) has a unique solution, it follows that *the following two properties are satisfied*

$$(4.1) \quad 1^\circ \quad \int (A_{j_1 \dots j_s}^{i_1 \dots i_r} + B_{j_1 \dots j_s}^{i_1 \dots i_r}) dx^k = \int A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k + \int B_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k$$

and

$$(4.2) \quad 2^\circ \quad \int C A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k = C \int A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k$$

where $C = \text{const.}$

$$(4.3) \quad 3^\circ \quad \nabla_{i(t)} \left(\int A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k \right) = A_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{dx^k}{dt}$$

PROOF. We shall prove it for $r=1$ and $s=2$. The general case can then be proved in the same way. (4.3) means

$$\begin{aligned} \frac{d}{dt} \left(\int A_{j_l}^i dx^k \right) + \left(\Gamma_{\lambda \mu}^k \int A_{j_l}^i dx^\lambda \right) \frac{dx^\mu}{dt} + \left(\Gamma_{\lambda \mu}^i \int A_{j_l}^\lambda dx^k \right) \frac{dx^\mu}{dt} - \\ - \left(\Gamma_{j \mu}^\lambda \int A_{\lambda l}^i dx^k \right) \frac{dx^\mu}{dt} - \left(\Gamma_{l \mu}^\lambda \int A_{j \lambda}^i dx^k \right) \frac{dx^\mu}{dt} = A_{j_l}^i \frac{dx^k}{dt}. \end{aligned}$$

In order to prove this we consider the equation

$$\begin{aligned} \int_{\tau(0)}^{\tau(t)} A_{j_l}^i dx^k = \int_{\tau(0)}^{\tau(t)} A_{j_l}^i dx^k - \int_{\tau(0)}^{\tau(t)} \Gamma_{rs}^k \left(\int_{\tau(0)}^{\tau(\theta)} A_{j_l}^i dx^r \right) dx^s - \\ - \int_{\tau(0)}^{\tau(t)} \Gamma_{rs}^i \left(\int_{\tau(0)}^{\tau(\theta)} A_{j_l}^r dx^k \right) dx^s + \int_{\tau(0)}^{\tau(t)} \Gamma_{js}^r \left(\int_{\tau(0)}^{\tau(\theta)} A_{rl}^i dx^k \right) dx^s + \int_{\tau(0)}^{\tau(t)} \Gamma_{ls}^r \left(\int_{\tau(0)}^{\tau(\theta)} A_{jr}^i dx^k \right) dx^s \end{aligned}$$

which is a special case of (2.5). Differentiating by t we obtain

$$\begin{aligned} \frac{d}{dt} \int A_{j_l}^i dx^k = A_{j_l}^i \frac{dx^k}{dt} - \Gamma_{rs}^k \left(\int A_{j_l}^i dx^r \right) \frac{dx^s}{dt} - \Gamma_{rs}^i \left(\int A_{j_l}^r dx^k \right) \frac{dx^s}{dt} + \\ + \Gamma_{js}^r \left(\int A_{rl}^i dx^k \right) \frac{dx^s}{dt} + \Gamma_{ls}^r \left(\int A_{jr}^i dx^k \right) \frac{dx^s}{dt} \end{aligned}$$

which was to be proved. ■

4° The equality

$$(4.4) \quad \int_{\tau(0)}^{\tau(t)} (\nabla_{\dot{\tau}(t)} A_{j_1 \dots j_s}^{i_1 \dots i_r}) dx^k = A_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{dx^k}{dt} + T_{j_1 \dots j_s}^{i_1 \dots i_r k}$$

holds, where T is a tensor field parallel along the curve $\tau(t)$, if and only if $\tau(t)$ is a geodesic line.

PROOF. As in 3° we shall assume that $r=1$ and $s=2$. Using the property 3°, from (4.4) we obtain

$$\begin{aligned} \nabla_{\dot{\tau}(t)} T_{jl}^i &= \nabla_{\dot{\tau}(t)} \left[\int_{\tau(0)}^{\tau(t)} (\nabla_{\dot{\tau}(t)} A_{jl}^i) dx^k - A_{jl}^i \frac{dx^k}{dt} \right] = \\ &= (\nabla_{\dot{\tau}(t)} A_{jl}^i) \frac{dx^k}{dt} - (\nabla_{\dot{\tau}(t)} A_{jl}^i) \frac{dx^k}{dt} - A_{jl}^i \nabla_{\dot{\tau}(t)} \frac{dx^k}{dt} = -A_{jl}^i \nabla_{\dot{\tau}(t)} \dot{\tau}(t) \end{aligned}$$

We see that the tensor field T is parallel if and only if $\nabla_{\dot{\tau}(t)} \dot{\tau}(t) = 0$, i.e. $\tau(t)$ is a geodesic line. ■

5° If the affine connection Γ admits parallel metric field g_{ij} (torsion tensor may not be equal to zero), then the metric tensor behaves as a constant for contravariant integration satisfying property 2°.

This property is an immediate consequence of 3°.

6° Contravariant integration commutes with the contraction for any of the indices $i_1, \dots, i_r, k$ with $j_1, \dots, j_s$.

This can be seen directly from the (2.5).

7° Let the tensor field $A_{j_1 \dots j_s}^{i_1 \dots i_r}$ be symmetric (antisymmetric) with respect to two lower or upper indices. Then $\int A_{j_1 \dots j_s}^{i_1 \dots i_r} dx^k$ is also symmetric (antisymmetric) with respect to the same indices.

PROOF. For example, let A_{ij} be an antisymmetric tensor field. Adding the equations

$$\begin{aligned} \int A_{ij} dx^k &= \int A_{ij} dx^k - \int \Gamma_{\lambda s}^k \left(\int A_{ij} dx^\lambda \right) dx^s + \\ &+ \int \Gamma_{is}^\lambda \left(\int A_{\lambda j} dx^k \right) dx^s + \int \Gamma_{js}^\lambda \left(\int A_{i\lambda} dx^k \right) dx^s \end{aligned}$$

and

$$\begin{aligned} \int A_{ji} dx^k &= \int A_{ji} dx^k - \int \Gamma_{\lambda s}^k \left(\int A_{ji} dx^\lambda \right) dx^s + \\ &+ \int \Gamma_{js}^\lambda \left(\int A_{\lambda i} dx^k \right) dx^s + \int \Gamma_{is}^\lambda \left(\int A_{j\lambda} dx^k \right) dx^s \end{aligned}$$

we obtain the relation

$$(4.5) \quad y_{ij}^k = - \int \Gamma_{\lambda s}^k y_{ij}^\lambda dx^s + \int \Gamma_{is}^\lambda y_{\lambda j}^k dx^s + \int \Gamma_{js}^\lambda y_{i\lambda}^k dx^s,$$

where

$$y_{ij}^k = \int A_{ij} dx^k + \int A_{ji} dx^k$$

The system (4.5) can be considered as a Volterra system of integral equations for y_{ij}^k . Thus the solution of (4.5) is unique. However $y_{ij}^k = 0$ is also a solution. Therefore

$$\int A_{ij} dx^k = - \int A_{ji} dx^k. \quad \blacksquare$$

5. Geometrical interpretation

Independently from the previous results we shall now introduce an integral operator independent from the choice of the coordinate system and we shall prove that it is the same operator which we have introduced earlier.

Let A be a tensor field of type (r, s) along the smooth curve τ . At the points $\tau(0), \tau(t/m), \tau(2t/m), \dots, \tau((m-1)t/m)$ and $\tau(t)$ we consider the tangent vectors to the curve τ

$$X_0 = \frac{t}{m} \dot{\tau}(0) \in T_{\tau(0)}(M), \quad X_1 = \frac{t}{m} \dot{\tau}\left(\frac{t}{m}\right) \in T_{\tau(t/m)}(M),$$

$$\dots X_{m-1} = \frac{t}{m} \dot{\tau}\left((m-1)\frac{t}{m}\right) \in T_{\tau((m-1)t/m)}(M) \quad \text{and} \quad X_m = \frac{t}{m} \dot{\tau}(t) \in T_{\tau(t)}(M)$$

Let $\varphi_{\tau(b)}^{\tau(a)}$ denote a parallel displacement from the point $\tau(a)$ to the point $\tau(b)$ along the curve τ . It can be proved that there exists the following limit

$$(5.1) \quad \lim_{m \rightarrow \infty} [\varphi_{\tau(t)}^{\tau(0)} A \otimes X_0 + \varphi_{\tau(t)}^{\tau(t/m)} A \otimes X_1 + \dots + \varphi_{\tau(t)}^{\tau((m-1)t/m)} A \otimes X_{m-1} + A \otimes X_m]$$

In fact the term in brackets is a tensor of type $(r+1, s)$ at the point $\tau(t)$, and so the limit (5.1) defines a tensor $B_{j_1 \dots j_s}^{i_1 \dots i_{r+1}}(\tau(t))$ of type $(r+1, s)$ at the point $\tau(t)$.

Now we shall prove the following equality

$$(5.2) \quad \hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} dx^k = B_{j_1 \dots j_s}^{i_1 \dots i_{r+1}}(\tau(t)).$$

This equality is obviously satisfied for $t=0$. So it is sufficient to prove that

$$\nabla_{\dot{\tau}(t)} \hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} dx^k = \nabla_{\dot{\tau}(t)} B_{j_1 \dots j_s}^{i_1 \dots i_{r+1}}(\tau(t)).$$

From (5.1) and (4.3) we obtain

$$\begin{aligned} \nabla_{\dot{\tau}(t)} B_{j_1 \dots j_s}^{i_1 \dots i_{r+1}}(\tau(t)) &= \left\{ \lim_{m \rightarrow \infty} \left[\varphi_{\tau(t)}^{\tau((m+1)/m)} B\left(\tau\left(\frac{t(m+1)}{m}\right)\right) - B(\tau(t)) \right] \frac{m}{t} \right\}_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} = \\ &= \left\{ \lim_{m \rightarrow \infty} [\varphi_{\tau(t)}^{\tau((m+1)/m)} (\varphi_{\tau(t(m+1)/m)}^{\tau(0)} A \otimes X_0 + \varphi_{\tau(t(m+1)/m)}^{\tau(t/m)} A \otimes X_1 + \dots \right. \\ &\quad \left. + \varphi_{\tau(t(m+1)/m)}^{\tau(t)} A \otimes X_m + A \otimes X_{m+1}) - (\varphi_{\tau(t)}^{\tau(0)} A \otimes X_0 + \right. \\ &\quad \left. + \varphi_{\tau(t)}^{\tau(t/m)} A \otimes X_1 + \dots + A \otimes X_m)] \frac{m}{t} \right\}_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} = \\ &= \left\{ \lim_{m \rightarrow \infty} \varphi_{\tau(t)}^{\tau((m+1)/m)} (A \otimes X_{m+1}) \frac{m}{t} \right\}_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} = \\ &= (A \otimes \dot{\tau}(t))_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} = A_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} \frac{dx^k}{dt} = \nabla_{\dot{\tau}(t)} \hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1 \dots j_s}^{i_1 \dots i_{r+1}} dx^k \end{aligned}$$

and the proof is finished.

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