

On π -t groups*

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In this paper we deal with t -groups, and we introduce the concept of π -t group. We generalize the results concerning solvable t -groups for the case of π -solvable π -t groups.

We recall the following definitions and theorems.

Definition. A t -group is a group G whose subnormal subgroups are all normal in G .

Definition. We say that a subgroup H of a group G is pronormal in G if and only if for all x in G , H and H^x are conjugate in $\langle H, H^x \rangle$.

The following result is well known:

Lemma A. A subgroup H of G is normal in G if and only if it is both subnormal and pronormal in G .

Lemma B. (PENG [1]). Let M be a normal p' -subgroup of G and let P be any p -subgroup of G . Then P is pronormal in G if and only if PM/M is pronormal in G/M . In 1957 GASCHÜTZ [2] proved the following

Theorem A. The subgroups of solvable t -groups are again t -groups. In 1969 PENG [1] showed.

Theorem B. G is a solvable t -group if and only if all subgroups of prime power order of G are pronormal in G .

In 1977 ASAAD, M. [3] proved the following

Theorem C. If each subgroup of prime power order of G' is pronormal in G and G' is a π -Hall subgroup of G , then G is a solvable t -group.

The converse of this theorem is not true. ASAAD gave the following example: $G = A \times B$, where A is a quaternion group and B is an abelian group of odd order. This theorem may be stated in the following more general form:

Theorem D. (PENG, T.: Personal communication)

Let N be a normal π -Hall subgroup of G . If G/N is a solvable t -group, and each subgroup of prime power order of N is pronormal in G , then G is a solvable t -group.

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We prove that if the order of the group G is odd, then the converse implication of Asaad's Theorem also holds.

Theorem 1. *Let G be a finite group of odd order. G is a solvable t -group if and only if each subgroup of prime power order of G' is pronormal in G , and G' is a π -Hall subgroup in G .*

For the proof we need the following Lemma.

Lemma 1. *Let G be a finite group, $G' < G$, $P \in \text{Syl } p(G)$, $p \in \pi(G')$, $p \neq 2$. Suppose, that each subnormal p -subgroup of $N_G(P)$ is normal in $N_G(P)$. Then either $P \leq G'$ or $C_G(P) = N_G(P)$ or both.*

PROOF. Obviously P is a t -group. As $p \neq 2$, it is well known, that P is abelian. Assume, that $P \not\leq G'$. Thus $PG' \triangleleft G$, whence $N_G(P)G' = G$ follows by a Frattini argument.

The Schur-Zassenhaus' theorem implies that there exists a subgroup K such that $N_G(P) = P \cdot K$, $P \cap K = 1$. Denote $P \cap G' = S$. Clearly $S \neq 1$. Obviously each element of K induces an automorphism of P by conjugation, and clearly they induce the identity on P/S . Applying a theorem of GLAUBERMAN [4] $C_P(K)S = P$ follows. Denote $L = C_P(K)$. We distinguish two cases

$$\text{a) } \Omega_1(P) \leq L$$

In this case $K \leq C_G(\Omega_1(P))$. As P is abelian, $K \leq C_G(P)$ follows. Thus $N_G(P) = C_G(P)$.

$$\text{b) } \Omega_1(P) \not\leq L$$

There exists a subgroup T such that

$$\Omega_1(P) = [L \cap \Omega_1(P)] \times T$$

If $r \in T$, $r \neq e$, then $\langle r \rangle \triangleleft \triangleleft N_G(P)$. By assumptions $\langle r \rangle \triangleleft N_G(P)$ follows. Obviously $K \not\leq C_G(\langle r \rangle)$. Consider $u \in L \cap \Omega_1(P)$, $u \neq e$. As $|ur| = p$, $\langle ur \rangle \triangleleft \triangleleft N_G(P)$, $\langle ur \rangle \triangleleft N_G(P)$ by assumptions. Let $k \in K$ be, such that, $k \in C_G(u)$ but $k \notin C_G(r)$. Since $K \leq N_G(\langle ur \rangle)$, there exists a natural number m such that

$$(ur)^k = (ur)^m, \text{ where } 2 \leq m \leq p-1 \text{ (} k \notin C_G(ur)\text{)}$$

Thus $ur^k = u^m r^m$, whence $u^{m-1} = r^k (r^m)^{-1}$ follows. As $K < N_G(\langle r \rangle)$, $r^k = r^n$, where $2 \leq n \leq p-1$. So $u^{m-1} = r^{n-m}$, but $\langle u \rangle \cap \langle r \rangle = 1$, thus $u^{m-1} = e$, a contradiction.

Now we turn to the proof of our theorem.

PROOF OF THEOREM 1. i, Assume, that each subgroup of prime power order of G_r is pronormal in G , and G' is a π -Hall subgroup of G . Then G is a solvable t -group by Asaad's Theorem.

ii, Conversely, suppose that G is a solvable t -group. It is well known, that G is supersolvable, and all its Sylow-subgroups are abelian. So G' is nilpotent. Thus each subgroup of prime power order of G' is subnormal in G , whence it is normal in G by assumption.

Lemma A implies, that each subgroup of prime power order of G' is pronormal in G . Now we prove, that G' is a π -Hall subgroup in G . Assume, that there exists a

$Q \in \text{Syl}_q(G)$ such that $Q \not\leq G'$, $Q \cap G' = Q^* \neq 1$. Using Theorem A, by Lemma 1. $N_G(Q) = C_G(Q)$. It is easy to see, that $N_G(Q)G' = G$, thus $C_G(Q)G' = G$. As Q is abelian and G' is nilpotent, $G' \leq C_G(Q^*)$ follows. Thus $C_G(Q^*) = G$. It is easy to show, that there exists a subgroup T in G such that

$$G = TQ, \quad T \triangleleft G, \quad T \cong G' \quad \text{and} \quad T \cap Q = Q^*.$$

Since $C_G(Q^*) = G$ it follows by the Schur-Zassenhaus theorem that there exists a subgroup L in T such that $T = LQ^*$, and $L \triangleleft T$. It is easy to see, that $L \triangleleft G$ and $G = LQ$. As Q is abelian therefore $L \cong G'$. Thus $Q \cap G' = 1$ contrary to the assumption $Q^* \neq 1$.

We introduce now the concept of π -t group.

Definition. Let G be a finite group, $\pi \subset \pi(G)$. G is called a π -t group if and only if its subnormal π -subgroups are all normal in G .

We construct a π -solvable π -t group, which possesses a subgroup that is not a π -t group. Let G be a finite group with following properties:

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft G_3 = G, \quad G_2/G_1 \cong P, \quad P \in \text{Syl}_p(G)$$

$$G_3/G_2 \cong Q, \quad Q \in \text{Syl}_q(G), \quad G_1 \in \text{Syl}_r(G)$$

$P = P_0 \times P_1$ where $|P_0| = p$ and P_1 is abelian, $G_1 \cdot P_0$ is a one step nonabelian group.

$$G_1 P_1 = G_1 \times P_1, \quad P_0 Q = P_0 \times Q, \quad G_1 Q = G_1 \times Q$$

furthermore $P_1 \cdot Q$ is a one step nonabelian group.

Let $\pi = \{p, q\}$. It is easy to show, that G is a π -solvable π -t group. Let X be a subgroup of P , such that $X \not\leq P_0$, $X \not\leq P_1$, $|X| = p$. We have $X \triangleleft P \triangleleft PQ$. Suppose $X \triangleleft PQ$, $X = \langle ab \rangle$, where $a \in P_1$, $b \in P_0$, $a \neq e$, $b \neq e$. Consider $c \in Q$, $c \neq e$. Then $(ab)^c = a^*b$ where $a^* \in P_1$. On the other hand $(ab)^c = (ab)^k$ where $2 \leq k \leq p-1$ ($c \notin C_G(ab)$). Thus $a^*b = a^k b^k$, hence $(a^k)^{-1} a^* = b^{k-1}$ follows.

But $\langle a \rangle \cap \langle b \rangle = 1$, so $b^{k-1} = e$, a contradiction. Thus PQ is not a π -t group.

In the following we shall derive some properties of π -solvable π -t groups. First of all we generalize the result of Peng.

Theorem 2. Let G be a finite group, $\pi \subset \pi(G)$. Let N be a π -solvable normal subgroup of G such that $(|G:N|, p) = 1$ for all $p \in \pi \cap \pi(N)$. If G/N is a π -t group, and each p -subgroup of N , where $p \in \pi$ is pronormal in G , then G is a π -t group.

For the proof we need the following

Lemma 2. Let U be a finite group, $\pi \subset \pi(U)$. Let L be a solvable π -subgroup of U and $L \triangleleft \triangleleft U$. Suppose that each p -subgroup of L , is pronormal in U . Then $L \triangleleft U$.

PROOF. We argue by induction on $|u|$. The solvability of L implies, that there exists a p -subgroup S $|p \in \pi|$ of L such that $S \triangleleft L$. We have $S \triangleleft \triangleleft U$ and S is pronormal in U , hence $S \triangleleft U$ follows by Lemma A. Set $\bar{U} = U/S$, $\bar{L} = L/S$. $\bar{L} \triangleleft \triangleleft \bar{U}$ holds. With help of Lemma B it is easy to show that each p -subgroup of $\bar{L}$ is pronormal in $\bar{U}$. As $|\bar{U}| < |U|$, $\bar{L} \triangleleft \bar{U}$ follows by induction. Thus $L \triangleleft U$.

Now we turn to the proof of our theorem.

PROOF OF THEOREM 2. Let H be a π -subgroup of G , such that $H \triangleleft \triangleleft G$. Obviously $H \cap N \triangleleft \triangleleft N$. Apply Lemma 2. to N and $H \cap N$, we obtain $H \cap N \triangleleft N$. As $H \cap N \triangleleft N \triangleleft G$, using Lemma 2. to G and $H \cap N$, we get $H \cap N \triangleleft G$. $(|H \cap N|, |H : H \cap N|) = 1$ by the conditions of the Theorem. The Schur—Zassenhaus theorem implies, that there exists a subgroup K in H such that $H = (H \cap N)K$ and $(H \cap N) \cap K = 1$. Clearly $H \cap N \triangleleft NK$, $H \triangleleft \triangleleft NK$. Obviously $(|K|, |N|) = 1$. Using Sylow's theorems it is very easy to see that $H \triangleleft NK$. We have

$$HN/N \cong H/H \cap N \cong K$$

As $H \triangleleft \triangleleft G$, therefore $NH/H \triangleleft \triangleleft G/N$. G/N is a π -t group by assumption, hence $HN/N \triangleleft G/N$, so $HN \triangleleft G$ follows. Thus $H \triangleleft NK \triangleleft G$. As $H \cap N \triangleleft G$, K is a Hall subgroup in NK , using Sylow's theorems it is easy to see that $H \triangleleft G$.

Theorem 3. Let G be a π -solvable finite group of odd order, where $\pi \subset \pi(G)$. G is a π -t group such that each subgroup of G is again a π -t group if and only if each p -subgroup of G' , where $p \in \pi$, is pronormal in G and $(|G : G'|, p) = 1$ for all $p \in \pi \cap \pi(G')$.

PROOF OF THEOREM 3. First, suppose that G' satisfies the above conditions. Theorem 2. is applicable with $N = G'$, thus we obtain that G is a π -t group. Let L be an arbitrary subgroup of G . We have

$$L/L \cap G' \cong LG'/G'.$$

As $LG'/G' \cong G/G'$, $L/L \cap G'$ is abelian. Clearly $(|L : L \cap G'|, p) = 1$ for all $p \in \pi$. Obviously each p -subgroup of $L \cap G'$ — where $p \in \pi$ — is pronormal in L . Applying Theorem 2. to L and $L \cap G'$, we obtain that L is a π -t group.

Conversely, assume that G is a π -solvable π -t group such that, each subgroup is a π -t group again.

a) Set $S \leq G'$, $|S| = p^k$, $p \in \pi$.

We show that S is pronormal in G . Let $x \in G$ be arbitrary. Consider $T = \langle S, S^x \rangle$. If $S \in \text{Syl}_p(T)$, then $S^x = S^u$, where $u \in T$. Thus we can assume that $S < P^*$ and $P^* \in \text{Syl}_p(T)$. Obviously there exists a $t \in T$ such that $S^{xt} < P^*$, $P^* \leq P \in \text{Syl}_p(G)$. We apply now the Theorem of Alperin [5] for G with S and S^{xt} . Thus there exist elements x_i and p -Sylow subgroups Q_i of G , $1 \leq i \leq n$, furthermore an element y of $N_G(P)$ which satisfy the following conditions:

- (i) $xt = x_1 x_2 \dots x_n y$
- (ii) $P \cap Q_i$ is a tame intersection, $1 \leq i \leq n$.
- (iii) x_i is a p -element of $N_G(P \cap Q_i)$, $1 \leq i \leq n$
- (iv) $S \leq P \cap Q_1$, while $S^{x_1 \dots x_i} \leq P \cap Q_{i+1}$, where $1 \leq i \leq n-1$.

Clearly $S \triangleleft \triangleleft P \cap Q_1 \triangleleft N_G(P \cap Q_1)$. By assumptions $N_G(P \cap Q_1)$ is a π -t group, hence $S \triangleleft N_G(P \cap Q_1)$. As $x_1 \in N_G(P \cap Q_1)$, $S = S^{x_1}$. Similarly we obtain $S^{x_1 \dots x_n} = S$. Thus $S^{xt} = S^y$, where $y \in N_G(P)$. Obviously $S \triangleleft \triangleleft P \triangleleft N_G(P)$. Since $N_G(P)$ is a π -t group, it follows that $S \triangleleft N_G(P)$. So $S^{xt} = S$. Thus $S^x = S^{t^{-1}}$ and S is pronormal in G .

b) Let $r \in \pi(G')$, $R \in \text{Syl}_r(G)$. Assume that $R \not\leq G'$, then $N_G(R) = C_G(R)$ by Lemma 1.

R is a t -group of odd order, so R is abelian. As $N_G(R) = C_G(R)$, G has a normal r -complement K by the Theorem of Burnside. Thus $RK = G$, $R \cap K = 1$, $K \triangleleft G$. Clearly $G/K \cong R$. As R is abelian, $G' \leq K$, whence $G' \cap R = 1$ follows, a contradiction. The proof is complete.

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