

Group rings of existentially closed locally finite p -groups

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Let K be any field. Assume that the group G is either existentially closed (e.c.) in the class of all groups, or e.c. in the class of all locally finite groups.*) Then a result of K. BONVALLET, B. HARTLEY, D. S. PASSMAN and M. K. SMITH [1] asserts that the group ring KG contains precisely one proper ideal, namely its augmentation ideal $\omega(KG)$. Obviously, a necessary ingredient for this result is the simplicity of G .

Now, consider the e.c. structures in the class $L\mathfrak{F}_p$ of all locally finite p -groups. Up to isomorphism, there exists a unique countable e.c. $L\mathfrak{F}_p$ -group E_p [5, Satz 2]. This group cannot be simple, since minimal normal subgroups of $L\mathfrak{F}_p$ -groups are cyclic of order p [2, 1.B.8]. However, E_p is characteristically simple [5, Satz 7]. And B. MAIER observed in [5, pp. 124—125] that the characteristical simplicity of E_p together with the main part of the proof of BONVALLET, HARTLEY, PASSMAN and SMITH still yields the following result: If K is a field with $\text{Char } K \neq p$, then $\omega(KE_p)$ is the unique proper ideal in KE_p which is invariant under the basis transformations of KE_p induced by the automorphisms of E_p .

It is the goal of the present note, to extend this result to the modular group rings of E_p .

Theorem. *Let K be any field, and let G be an e.c. $L\mathfrak{F}_p$ -group.*

(a) *For every proper ideal I of KG , there exists a nontrivial normal subgroup N in G such that $\omega(KN) \cdot KG \subseteq I$.*

(b) *If G is countable, then $\omega(KG)$ is the unique proper ideal in KG which is invariant under the basis transformations of KG induced by the automorphisms of G . In particular, KG is either characteristically simple, or $\omega(KG)$ is the unique proper characteristic ideal in KG .*

Clearly, every ideal of $\omega(KG)$ is also an ideal in KG . Hence, if $\omega(KG)$ is the unique proper characteristic ideal in KG , then $\omega(KG)$ is characteristically simple.

Note also, that uncountable e.c. $L\mathfrak{F}_p$ -groups are in general not characteristically simple by a result of S. THOMAS [6, Theorem 1]. Therefore, it seems to be unlikely that part (b) of the Theorem does also hold for e.c. $L\mathfrak{F}_p$ -groups of arbitrary cardinality.

In the proof of the Theorem, we will use wreath product constructions. Recall that, for any two groups A and B , the unrestricted regular wreath product $W = AWr B$

* Cf. [3], [5] or [6] for the definition of "e.c.".

is the split extension $W = \Omega \rtimes B$ of its base group $\Omega = \{f \mid f: B \rightarrow A\} = A^B$ (with componentwise multiplication) by its top group B with regard to the action $(b')f^b = (b'b^{-1})f$ for all $f \in \Omega$ and $b, b' \in B$. The support of a function $f \in \Omega$ is defined as

$$\text{supp } f = \{b \in B \mid (b)f \neq 1\}.$$

We will always identify in the natural way B with the top group of W , and A with the 1-component $\{f \in \Omega \mid \text{supp } f \subseteq \{1\}\}$ of W . Note, that

$$\text{supp } a^b \subseteq \{b\} \quad \text{for all } a \in A, b \in B.$$

Proof of the Theorem. Part (b) follows from (a), since every countable e.c. $L\mathfrak{F}_p$ -group is characteristically simple [5, Satz 7], and since $\omega(KG)$ has codimension 1 in KG .

For the proof of part (a), suppose we are given a finite elementary abelian p -group A , and a finite subgroup X of G . Then $AWrX = \langle A, X \rangle$ and G are contained in the $L\mathfrak{F}_p$ -group $AWrG$. And since G is e.c. in $L\mathfrak{F}_p$, we obtain an embedding

$$\sigma: AWrX \rightarrow G \quad \text{with} \quad \sigma \upharpoonright X = id.$$

(The group table of $AWrX$ (with the elements from X as constants) can be duplicated in G .) This shows, that we can follow the proof of the theorem of BONVALLET, HARTLEY, PASSMAN and SMITH [1], in order to establish part (a) whenever $\text{Char } K \neq p$, as observed by B. MAIER.

Now, assume that $\text{Char } K = p$. Let $0 \neq \alpha \in I$. Then there exist pairwise distinct elements $x_0, \dots, x_n \in G$ such that

$$\alpha = \sum_{i=0}^n k_i x_i^{-1} \quad \text{with} \quad 0 \neq k_i \in K \quad \text{for} \quad 0 \leq i \leq n.$$

Since I is an ideal in KG , we may assume without loss that $x_0 = 1$ and $k_0 = 1$. Choose $m \in \mathbb{N}$ such that $p^m \equiv 2^n - 1$. Put $X = \langle x_1, \dots, x_n \rangle$ and

$$Y = \langle y_1 \rangle X \dots X \langle y_n \rangle \quad \text{where each } \langle y_i \rangle \text{ is cyclic of order } p^{m+1}.$$

Arguing as above, we can find an embedding

$$\sigma: YWrX \rightarrow G \quad \text{with} \quad \sigma \upharpoonright X = id.$$

Therefore, we may assume without loss that $X \leq YWrX \leq G$. Put

$$z_i = [y_i, x_i] \quad \text{for} \quad 1 \leq i \leq n,$$

and observe that z_i is an element of order p^{m+1} in the base group Θ of $YWrX$.

By inverse induction over $s \in \{n, n-1, \dots, 0\}$, we will now find elements

$$\alpha_s = \sum_{i=0}^s \alpha_{s,i} \cdot x_i^{-1} \in I$$

with

$$\alpha_{s,i} \in K\Theta \quad \text{and} \quad \alpha_{s,0} = (z_n - 1)(z_{n-1} - 1) \dots (z_{s+1} - 1).$$

To this end, we proceed as follows. Start with $\alpha_n = \alpha$; then $\alpha_{n,i} = k_i \in K$ and $\alpha_{n,0} = k_0 = 1$. Now suppose, that α_s has been found for some $s \geq 1$. Put

$$\alpha_{s-1} = z_s \cdot \alpha_s - y_s^{-1} \cdot \alpha_s \cdot y_s \in I.$$

Since $K\Theta$ is commutative, we have

$$\alpha_{s-1} = \sum_{i=0}^s \alpha_{s,i} \cdot (z_s - [y_s, x_i]) \cdot x_i^{-1}.$$

Obviously,

$$\alpha_{s-1,i} = \alpha_{s,i} \cdot (z_s - [y_s, x_i]) \in K\Theta \quad \text{and}$$

$$\alpha_{s-1,0} = \alpha_{s,0} \cdot (z_s - 1) = (z_n - 1)(z_{n-1} - 1) \dots (z_s - 1).$$

Moreover, $\alpha_{s-1,s} = 0$ by the definition of z_s . Thus, the induction is completed, and we obtain

$$\alpha_0 = \alpha_{0,0} \cdot 1 = (z_n - 1)(z_{n-1} - 1) \dots (z_1 - 1) \in I.$$

Because of $\text{supp } z_i = \{1, x_i\}$, we have

$$\langle z_1, \dots, z_n \rangle = \langle z_1 \rangle \times \dots \times \langle z_n \rangle$$

where each $\langle z_i \rangle$ is cyclic of order p^{m+1} .

Put $J = \{1, \dots, n\}$. Define

$$u_S = \prod_{j \in S} z_j^{-1} \quad \text{for every } S \subseteq J.$$

Then

$$\alpha_0 = \sum_{S \subseteq J} \varepsilon_S \cdot u_S^{-1} \in I \quad \text{where } \varepsilon_S = (-1)^{|J \setminus S|}.$$

Note, that $\langle u_S \rangle$ is cyclic of order p^{m+1} whenever $S \neq \emptyset$. Let

$$U = \langle u_S | S \subseteq J \rangle = \langle z_1 \rangle \times \dots \times \langle z_n \rangle,$$

and define

$$V = \prod_{\substack{S \subseteq J \\ S \neq \emptyset}} \langle v_S \rangle \quad \text{where each } \langle v_S \rangle \text{ is cyclic of order } p^{m+2}.$$

If $W = VWrU$, then every

$$w_S = \left[\prod_{r=0}^{p^{m+1}-2} v_S^{(u_S^r)} \right]^{-1} \cdot \left[v_S^{(u_S^{p^{m+1}-1})} \right]^{p^{m+1}-1}, \quad \emptyset \neq S \subseteq J,$$

is contained in the base group Ω of W .

Let T_S be a complement to $\langle u_S \rangle$ in U . Then

$$\langle w_S^t | t \in T_S \rangle = \prod_{t \in T_S} \langle w_S^t \rangle,$$

since $\text{supp } w_S^t \subseteq \langle u_S \rangle \cdot t$. Further, $\langle w_S^t | t \in T_S \rangle \subseteq \langle v_S^U \rangle$, while V is the direct product of the $\langle v_S \rangle$, $\emptyset \neq S \subseteq J$. Hence,

$$\Omega_0 = \langle w_S^t | t \in T_S, \emptyset \neq S \subseteq J \rangle = \prod_{\substack{S \subseteq J \\ S \neq \emptyset}} \prod_{t \in T_S} \langle w_S^t \rangle.$$

Define

$$L = \langle (w_{S_1}^p)^{t_1} \cdot (w_{S_2}^p)^{t_2} \mid t_1 \in T_{S_1}, t_2 \in T_{S_2}, \emptyset \neq S_1, S_2 \subseteq J \rangle.$$

Now, $p \cdot (p^{m+1} - 1) \equiv -p \pmod{p^{m+2}}$, and therefore

$$w_S^p = \left[\prod_{r=0}^{p^{m+1}-1} v_S^{(u_S^r)} \right]^{-p}.$$

Thus, u_S centralizes w_S^p . And since $U = \langle u_S \rangle \times T_S$, the set $\{(w_S^p)^t \mid t \in T_S\}$ is U -invariant for every non-empty $S \subseteq J$. Hence, L is a normal subgroup of W . Note, that Ω_0/L is the central product of the groups $\langle w_S^t \rangle$, $t \in T_S$, $\emptyset \neq S \subseteq J$, with amalgamations $(w_{S_1}^p)^{t_1} = (w_{S_2}^p)^{t_2}$.

Let $\theta: W \rightarrow W/L$ be the canonical epimorphism. Because of $U \cap L \leq U \cap \Omega = 1$, the restriction $\theta|_U$ is an embedding. If

$$\Sigma: 1 = U_0\theta < U_1\theta < \dots < U_l\theta = U\theta$$

is any chief series in $U\theta$, then the terms of Σ are exactly the intersections of $U\theta$ with the terms of the normal series

$$1 < \Omega/L < \Omega U_1/L < \dots < \Omega U_l/L = W/L$$

in W/L . Therefore, [3, Theorem 2.1] ensures the existence of an $L\mathfrak{F}_p$ -group $H \cong G$ and an embedding $\tau: W/L \rightarrow H$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{id} & G \\ \theta \downarrow & & \downarrow id \\ W/L & \xrightarrow{\tau} & H \end{array}$$

commutes. Since G is e.c. in $L\mathfrak{F}_p$, there does already exist an embedding

$$\hat{\tau}: W/L \rightarrow G \quad \text{with} \quad (\theta\hat{\tau})|_U = id.$$

Identifying U via θ with $U\theta$, and suppressing $\hat{\tau}$, we may assume without loss that $U = U\theta \leq W/L \leq G$. In the sequel, epimorphic images modulo L will be denoted by bars.

Let $\pi: \{0, \dots, 2^n - 1\} \rightarrow \{S \mid S \subseteq J\}$ be a bijection with $0\pi = \emptyset$. By inverse induction over $s \in \{2^n - 1, 2^n - 2, \dots, 0\}$, we will find now elements

$$\beta_s = \sum_{i=0}^s \beta_{s,i} \cdot \bar{u}_{i\pi}^{-1} \in I \quad \text{with}$$

$$\beta_{s,i} \in K\bar{\Omega} \quad \text{and} \quad \beta_{s,0} = \varepsilon_s \cdot \prod_{j=s+1}^{2^n-1} (\bar{w}_{j\pi} - 1).$$

To this end, we proceed as follows. Start with $\beta_{2^n-1} = \alpha_0$; then $\beta_{2^n-1,i} = \varepsilon_{i\pi} \in K$ and $\beta_{2^n-1,0} = \varepsilon_\emptyset$. Now suppose, that β_s has been found for some $s \geq 1$. Put

$$\gamma_s = \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{r+1} \in \Omega,$$

and let

$$\beta_{s-1} = \bar{w}_{s\pi} \cdot \beta_s - \bar{\gamma}_s^{-1} \cdot \beta_s \cdot \bar{\gamma}_s \in I.$$

Since $K\bar{\Omega}$ is commutative, we have

$$\beta_{s-1} = \sum_{i=0}^s \beta_{s,i} \cdot (\bar{w}_{s\pi} - [\bar{\gamma}_s, \bar{u}_{i\pi}]) \cdot \bar{u}_{i\pi}^{-1}.$$

Thus,

$$\beta_{s-1,i} = \beta_{s,i} \cdot (\bar{w}_{s\pi} - [\bar{\gamma}_s, \bar{u}_{i\pi}]) \in K\bar{\Omega}.$$

Further,

$$\begin{aligned} [\gamma_s, u_{s\pi}] &= \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{-r-1} \cdot \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^{r+1})}]^{r+1} = \\ &= \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{-r-1} \cdot \prod_{r=1}^{p^{m+1}-1} [v_{s\pi}^{(u_{s\pi}^r)}]^r = \\ &= \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{-1} \cdot [v_{s\pi}^{(u_{s\pi}^{p^{m+1}-1})}]^{p^{m+1}-1} = w_{s\pi}. \end{aligned}$$

Therefore, $\beta_{s-1,s}=0$. Moreover, $u_{0\pi}=u_\emptyset=1$, and thus

$$\beta_{s-1,0} = \beta_{s,\emptyset} \cdot (\bar{w}_{s\pi} - 1) = \varepsilon_\emptyset \cdot \prod_{j=s}^{2^n-1} (\bar{w}_{j\pi} - 1).$$

This completes the induction, and we obtain

$$\delta = \varepsilon_\emptyset \cdot \beta_0 = \varepsilon_\emptyset \cdot \beta_{0,0} = \prod_{\substack{S \subseteq J \\ S \neq \emptyset}} (\bar{w}_S - 1) \in I.$$

From the definition of L , we have $\bar{w}_{S_1}^p = \bar{w}_{S_2}^p$ for any two nonempty subsets S_1, S_2 of J . Thus, there exists $h \in G$ with $h = \bar{w}_S^p$ for $\emptyset \neq S \subseteq J$. Because of $\text{Char } K = p$, it follows that

$$\delta^p = \prod_{\substack{S \subseteq J \\ S \neq \emptyset}} (\bar{w}_S^p - 1) = (h - 1)^{2^n-1} \in I.$$

Since $p^m \geq 2^n - 1$, successive multiplication with $(h - 1)$ eventually yields

$$(h - 1)^{p^m} = (h^{p^m} - 1) \in I.$$

But h is an element of order p^{m+1} in $\Omega_0/L \cong G$. Therefore, $N = \{g \in G \mid (g - 1) \in I\}$ is a non-trivial normal subgroup of G with $\omega(KN) \cdot KG \subseteq I$. $\square$

Finally note, that the normal subgroups of every e.c. $L\mathfrak{F}_p$ -group are totally ordered by inclusion [3, Theorem 2.3], and that the embeddings of finite p -groups into e.c. $L\mathfrak{F}_p$ -groups are quite well understood (see [3, § 3]). In particular, in the proof of part (a) of the Theorem, if $f \in G$ is any fixed element with $\langle f^G \rangle \cap X = 1$, then a combination of [3, Theorem 3.1] and [4, Theorem 3.3] yields that the embeddings σ and $\hat{\tau}$ can be so chosen that, in addition,

$$\langle f^G \rangle \cap \text{Im } \sigma = 1 \quad \text{and} \quad \langle f^G \rangle \cap \text{Im } \hat{\tau} = 1.$$

But then, $f \in \langle f^G \rangle \cong \langle h^G \rangle \cong N = \{g \in G \mid (g - 1) \in I\}$. This shows, that the following stronger version of part (a) of the Theorem holds.

Proposition. *Let K be any field, and let G be an e.c. $L\mathfrak{F}_p$ -group. Assume that I is a proper ideal of KG , and that*

$$0 \neq \alpha = \sum_{i=0}^n k_i x_i \in I \quad \text{with} \quad 0 \neq k_i \in K$$

and pairwise distinct elements $1 = x_0, x_1, \dots, x_n \in G$. If $X = \langle x_1, \dots, x_n \rangle$, and if $\tilde{M}/\tilde{N}$ is the (unique) chief factor in G with $1 = \tilde{N} \cap X \not\subseteq \tilde{M} \cap X$, then $\tilde{N} \cong N = \{g \in G \mid (g-1) \in \in I\} \trianglelefteq G$ and $\omega(K\tilde{N}) \cdot KG \subseteq I$.

However, in the above situation, it remains open whether for every proper ideal I of KG there exists $N \trianglelefteq G$ such that

$$I \cap \omega(KG) = \omega(KN) \cdot KG,$$

i.e., whether the $\omega(KN) \cdot KG$, $N \trianglelefteq G$, are the only ideals of $\omega(KG)$.

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