

Characterization of additive functions with values in the circle group

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1. Let G be a metrically compact Abelian group, and T the one dimensional torus. A function $\varphi: \mathbf{N} \rightarrow G$ will be called completely additive, if $\varphi(mn) = \varphi(m) + \varphi(n)$ holds for each couple of $m, n \in \mathbf{N}$. Let $\mathcal{A}_G^*$ be the class of completely additive functions.

Let F be a strictly monotonically increasing function defined on $\mathbf{N}$ and taking positive integer values for each large integer, i.e. $F(n) \in \mathbf{N}$ if $n > n_0$. Let $\{x_n\}_{n=1}^\infty$ be an infinite sequence in G . We shall say that it is of property $D[F]$, if for any convergent subsequence $\{x_{v_n}\}_{n=1}^\infty$ the subsequence $\{x_{v_{F(n)}}\}_{n=1}^\infty$ has also a limit. We say that it is of property $\Delta[F]$ if $\{x_{F(n)} - x_n\}_{n=1}^\infty$ is convergent. It is clear, that if $\{x_n\} \in \Delta[F]$, then $\{x_n\} \in D[F]$. For the linear function $F(n) = an + b$, $a \in \mathbf{N}$, $b \in \mathbf{Z}$ we shall write $D[a, b]$, $\Delta[a, b]$ instead of $D[an + b]$, $\Delta[an + b]$.

Let $\mathcal{A}_G^*(D[F])$, $\mathcal{A}_G^*(\Delta[F])$ be the classes of those $\varphi \in \mathcal{A}_G^*$ for which $\{x_n = \varphi(n)\}_{n=1}^\infty$ is of property $D[F]$, $\Delta[F]$, respectively.

We are interested in giving a complete determination of $\mathcal{A}_G^*(D[F])$, $\mathcal{A}_G^*(\Delta[F])$.

We considered this problem with $F(n) = n + 1$ in some earlier papers [1–4]. Recently E. WIRSING [5] proved that $\varphi \in \mathcal{A}_T^*(D[0, 1])$ if and only if

$$(1.1) \quad \varphi(n) \equiv \tau \log n \pmod{1} \quad (n \in \mathbf{N})$$

for a $\tau \in \mathbf{R}$.

In [1] we proved that $\mathcal{A}_G^*(\Delta[0, 1]) = \mathcal{A}_G^*(D[0, 1])$, and by using WIRSING's theorem in [2] we deduced the following assertion: If $\varphi \in \mathcal{A}_G^*(\Delta[0, 1]) = \mathcal{A}_G^*(D[0, 1])$, then there exists a continuous homomorphism $\Psi: \mathbf{R}_x \rightarrow G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that φ is a restriction of Ψ on the set $\mathbf{N}$, i.e. $\varphi(n) = \Psi(n)$ ($\forall n \in \mathbf{N}$). The converse assertion is obvious. If $\Psi: \mathbf{R}_x \rightarrow G$ is a continuous homomorphism, then $\varphi(n) := \Psi(n) \in \mathcal{A}_G^*(\Delta[0, 1]) \subseteq \mathcal{A}_G^*(D[0, 1])$. These results have been extended for additive functions (omitting the completeness of additivity) in [3], [4].

The case $F(n) = n + b$ can be treated similarly as $F(n) = n + 1$. In this paper we shall investigate the case $F(n) = 2n - 1$. A complete determination of $\mathcal{A}_G^*(\Delta(2, -1))$ can be given easily, by using previous results [1], [2], [5] (see Section 4).

The characterization of $\mathcal{A}_G^*(D[2, -1])$ seems to be more complicated. We can solve it for $G = T$, (Theorem).

In the last section we shall formulate some conjectures.

2. Let $\varphi \in \mathcal{A}_G^*(D[2, -1])$. Let X denote the set of limit points of $\{\varphi(n) | n \in \mathbb{N}\}$, i.e. $g \in X$ if there exists $n_1 < n_2 < \dots, n_v \in \mathbb{N}$, for which $\varphi(n_v) \rightarrow g$. Let $\varphi(2n_v - 1) \rightarrow g'$. Then g' is determined by g . So the correspondence $L: g \rightarrow g'$ is a function. Let $X_0 (\subseteq X)$ be the set of limit points of $\{\varphi(2m+1) | m \in \mathbb{N}\}$. Then $L: X \rightarrow X_0, L[X] = X_0$, and as it is easy to see, L is a continuous function. It is clear that X and X_0 are closed semigroups in G , so by a known theorem (see [5], Theorem (9.16)) they are compact groups.

For the proof of these simple assertions see [1].

Since $0 \in X_0 \subset X$, we have $\varphi(n) \in X, \varphi(2n+1) \in X_0$ for each $n \in \mathbb{N}$.

Lemma 1. Let $S: X \rightarrow X_0$ be defined by

$$(2.1) \quad S(g) = L(2g + \varphi(2)) - L(g).$$

If $n_1 < n_2 < \dots, n_v \in \mathbb{N}$ is such a sequence for which $\varphi(n_v) \rightarrow g$, then $\varphi(2n_v + 1) \rightarrow S(g)$ ($v \rightarrow \infty$).

PROOF. Since $\varphi \in \mathcal{A}_G^*(D[2, -1])$, $\varphi(n_v) \rightarrow g$ implies that $\varphi(2n_v^2) \rightarrow \varphi(2) + 2g$,

$$\varphi(2 \cdot (2n_v^2) - 1) \rightarrow L(\varphi(2) + 2g),$$

$$\varphi(2n_v - 1) \rightarrow L(g), \quad \varphi(2n_v + 1) = \varphi(2 \cdot (2n_v^2) - 1) - \varphi(2n_v - 1) \rightarrow S(g). \quad \square$$

Corollary 1. We have

$$(2.2) \quad S(-\varphi(2)) = 0.$$

PROOF. Substitute $g = -\varphi(2)$ in (2.1). $\square$

Let $g \in X, n_1 < n_2 < \dots$ be such a sequence for which $\varphi(n_v) \rightarrow g$. Let $m_v = 2n_v, l_v^{(k)} = m_v^k + m_v^{k-1} + \dots + m_v + 1$. Then $\varphi(m_v) \rightarrow \varphi(2) + g, \varphi(l_v^{(1)}) \rightarrow S(g)$. We have $l_v^{(k)} = 1 + m_v l_v^{(k-1)} = 1 + 2n_v \cdot l_v^{(k-1)}$, and so the limits $\varphi(l_v^{(k)}) \rightarrow \lambda_k$ ($v \rightarrow \infty$) exist and

$$(2.3) \quad \lambda_k = S(g + \lambda_{k-1}), \quad k \geq 1, \quad \lambda_0 = 0.$$

Since $\varphi(2^{k-1} n_v^k) \rightarrow (k-1)\varphi(2) + kg$, we have

$$\varphi(m_v^k - 1) \rightarrow L((k-1)\varphi(2) + kg), \quad \text{and by } m_v^k - 1 = (m_v - 1) l_v^{(k-1)}$$

and we get

$$(2.4) \quad L((k-1)\varphi(2) + kg) = L(g) + \lambda_{k-1} \quad (k \geq 1)$$

Lemma 2. Let $E_0 = \{h | S(h) = 0\}$. Then E_0 is closed.

Furthermore, $h \in E_0$ if and only if

$$(2.5) \quad L((k-1)\varphi(2) + kh) = L(h), \quad \forall k \in \mathbb{Z}$$

$$(2.6) \quad L(-\varphi(2)) = L(h).$$

PROOF. It is clear that E_0 is closed. If (2.5) holds for $k=2$, then $S(h)=0, h \in E_0$. Let us assume that $h \in E_0$. Then (2.5) holds with $k=2$. Apply now (2.3), (2.4) with h instead of g . We have $\lambda_k = 0$ for each $k \geq 1$, and (2.4) gives (2.5) for positive integers k . Let $U_h = \{k(\varphi(2) + h) | k \in \mathbb{N}\}$. So we proved that

$$(2.6) \quad L(h) = L(u - \varphi(2)) \quad \forall u \in U_h.$$

Let $\bar{U}_h$ be the smallest closed set that contains U_h . Since L is a continuous function and U_h is a semigroup, U_h is a closed semigroup, therefore by the cited theorem in [5], we have that $\bar{U}_h$ is a compact group. Since $\varphi(2)+h \in U_h$, the whole cyclic group $\{k(\varphi(2)+h) | k \in \mathbb{Z}\} \subseteq \bar{U}_h$, consequently (2.5) holds for negative k 's as well. For $k=0$ we have (2.6). $\square$

Let

$$(2.7) \quad \underline{R} = \{R_1 < R_2 < \dots\}$$

be an arbitrary infinite sequence of positive integers. We shall say that $\underline{R}$ belongs to $\mathcal{P}_0$ if for each $d \in \mathbb{N}$, d divides R_n for every large n , i.e. if $v > v_0(\underline{R}, d)$. Let $\tilde{\mathcal{P}}_0 (\subseteq \mathcal{P}_0)$ be the set of those $\underline{R} \in \mathcal{P}_0$ for which $\lim_n \varphi(R_n)$ exists. For an arbitrary sequence $\underline{R}$ let

$$a(\underline{R}) = \lim_{v \rightarrow \infty} \varphi(R_v),$$

if the limit exists.

Let $a(\tilde{\mathcal{P}}_0)$ be the set of all limit points of a $(\underline{R})$, $\underline{R} \in \tilde{\mathcal{P}}_0$. If $\underline{R} \in \tilde{\mathcal{P}}_0$ and $d \in \mathbb{N}$, then $d\underline{R} = \{dR_1 < dR_2 < \dots\} \in \tilde{\mathcal{P}}_0$, and $a(d\underline{R}) = \varphi(d) + a(\underline{R})$. If $\underline{R} \in \tilde{\mathcal{P}}_0$, $\underline{S} = \{S_1 < S_2 < \dots\} \in \mathcal{P}_0$, then $\underline{R} * \underline{S} = \{R_1 S_1 < R_2 S_2 < \dots\} \in \tilde{\mathcal{P}}_0$, $a(\underline{R} * \underline{S}) = a(\underline{R}) + a(\underline{S})$. So the set $\{a(\underline{R}) | \underline{R} \in \tilde{\mathcal{P}}_0\}$ is a semigroup, the set of limit points is a semigroup as well, it is closed, so it is a compact group. Furthermore, from $a(d\underline{R}) = \varphi(d) + a(\underline{R})$ we have that $\varphi(d) \in a(\tilde{\mathcal{P}}_0)$, consequently $X \subseteq a(\tilde{\mathcal{P}}_0)$. The relation $a(\tilde{\mathcal{P}}_0) \subseteq X$ is obvious. So we have $X = a(\tilde{\mathcal{P}}_0)$.

Let $\underline{M} \in \tilde{\mathcal{P}}_0$, $a(\underline{M}) = g$. Let k be an arbitrary natural number. For each large v we have $M_v = 2kt_v$, $t_v \in \mathbb{N}$, $t_v < t_{v+1}$.

Now

$$\varphi(t_v) \rightarrow g - \varphi(2k) \quad (v \rightarrow \infty).$$

So we have

$$\varphi(M_v + k) = \varphi(2kt_v + k) = \varphi(k) + \varphi(2t_v + 1) \rightarrow \varphi(k) + S(g - \varphi(2k)),$$

and so

$$(2.8) \quad \varphi(M_v + 2m) \rightarrow \varphi(2m) + S(g - \varphi(4m))$$

$$(2.9) \quad \varphi(M_v + 2m + 1) \rightarrow \varphi(2m + 1) + S(g - \varphi(2m + 1))$$

$$(2.10) \quad \varphi(M_v + 2m - 1) \rightarrow \varphi(2m - 1) + S(g - \varphi(2(2m - 1))).$$

Now we observe the following relation. If $n_1 < n_2 < \dots$ is such a sequence of integers for which $\varphi(2n_v) \rightarrow \kappa$, then $\varphi(2n_v + 1) \rightarrow S(\kappa - \varphi(2))$. This follows immediately from the relation $\varphi(n_v) \rightarrow \kappa - \varphi(2)$.

Let us apply now this with $2n_v = M_v + 2m$. From (2.8), (2.9) we get that

$$(2.11) \quad S(\varphi(m) + S(g - \varphi(4m))) = \varphi(2m + 1) + S(g - \varphi(2(2m + 1))).$$

Substitute $g = \varphi(2m)$ in (2.10) and observe (2.2):

$$(2.12) \quad S(\varphi(m)) = \varphi(2m + 1) + S(\varphi(m) - \varphi(2m + 1)).$$

Since S is continuous, and (2.11) is true for every m , it is true for the limit points as well. Let $\tau \in X$, $\varphi(m_v) \rightarrow \tau$. Then $\varphi(2m_v + 1) \rightarrow S(\tau)$, and from (2.12),

$$S(\tau) = S(\tau) + S(\tau - S(\tau))$$

which implies that

$$(2.13) \quad S(\tau - S(\tau)) = 0 \quad \forall \tau \in X.$$

If $\varphi(2n_v) \rightarrow \kappa$, then $\varphi(n_v) \rightarrow \kappa - \varphi(2)$, and so $\varphi(2n_v - 1) \rightarrow L(\kappa - \varphi(2))$. Apply this with $2n_v = M_v + 2m$. From (2.9), (2.10) we get that

$$L(\varphi(m) + S(g - \varphi(4m))) = \varphi(2m - 1) + S(g - \varphi(2(2m - 1))).$$

Let $g = \varphi(2m)$. By (2.2) we have

$$L(\varphi(m)) = \varphi(2m - 1) + S(\varphi(m) - \varphi(2m - 1)),$$

and so by using the continuity of L and S , we get

$$(2.14) \quad S(\tau - L(\tau)) = 0 \quad \forall \tau \in X.$$

Let $g \in X$, $\varphi(n_v) \rightarrow g$. Then $\varphi(2n_v - 1) \rightarrow L(g)$, $\varphi(2n_v + 1) \rightarrow S(g)$,

$$S(L(g)) \leftarrow \varphi(2(2n_v - 1) + 1) = \varphi(4n_v - 1) \rightarrow L(\varphi(2) + g),$$

$$L(S(g)) \leftarrow \varphi(2(2n_v + 1) - 1) = \varphi(4n_v + 1) \rightarrow S(\varphi(2) + g).$$

So we have proved

Lemma 3. *We have*

$$(2.15) \quad SL(g) = L(\varphi(2) + g),$$

$$(2.16) \quad LS(g) = S(\varphi(2) + g),$$

$\forall g \in X$.

Lemma 4. *Assume that $S(g) = L(g) \quad \forall g \in X_0$. Then*

$$(2.17) \quad S(g) = L(g) \quad \forall g \in X.$$

PROOF. Let $Y_k = k\varphi(2) + X_0$ ($k = 0, 1, 2, \dots$). We shall prove that (2.17) holds for $g \in Y_k$ ($k = 0, 1, \dots$).

We shall use the relations (2.15), (2.16). By the assumption this is true for $k = 0$. Let now $g \in Y_1$.

Then $g = \varphi(2) + h$ with a suitable $h \in Y_0$. So $S(h) = L(h)$.

Since $S(h) \in Y_0$, we have $L(S(h)) = S(L(h))$, i.e. our assertion is true for $k = 1$. Assume that (2.17) is proved for $g \in Y_0 \cup Y_1 \cup \dots \cup Y_{k-1}$. Let $g \in Y_k$. Then $g = \varphi(2) + h$, $h \in Y_{k-1}$. Then $L(h) = S(h) \in Y_0$, and so $SL(h) = LS(h)$, and so (2.17) is true for each $g \in Y_k$. Since $\bigcup Y_k$ is everywhere dense in X , and S, G are continuous functions, therefore (2.17) is true for each $g \in X$. $\square$

Lemma 5. *Let $h \in X$. We have $L(h) = 0$ if and only if $S(h) = 0$.*

PROOF. Assume that $L(h) = 0$. Then, by (2.14) we have $S(h) = 0$. It is clear that there exists at least one h^* , such that $L(h^*) = 0$. Then $h^* \in E_0$, and so by (2.6), $L(-\varphi(2)) = L(h^*) = 0$. So we have $L(-\varphi(2)) = 0$. Let now $S(h) = 0$, i.e. $h \in E_0$. Then by (2.6) we have $L(h) = 0$.

Lemma 6. *Let V_g denote the closure of the set $\{kg | k \in \mathbb{N}\}$, $g \in X$. Then V_g is a compact subgroup of X . If $h \in E_0$, then $-\varphi(2) + V_{h+\varphi(2)} \subseteq E_0$.*

PROOF. The assertion that V_g is a compact subgroup is well-known. If $h \in E_0$, then by Lemma 2 and Lemma 5 we have $(k-1)\varphi(2) + kh = k(h + \varphi(2)) - \varphi(2) \in E_0$ for each $k \in \mathbb{Z}$. Since L, S are continuous, our assertion follows immediately. $\square$

Lemma 7. *If E_0 contains the only element $-\varphi(2)$, then $S(\tau) = L(\tau) = \tau + \varphi(2)$ $\tau \in X$, furthermore $\varphi(n+1) - \varphi(n) \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF. Since $E_0 = \{-\varphi(2)\}$, from (2.13), (2.14) we have $\tau - S(\tau) = -\varphi(2)$, $\tau - L(\tau) = -\varphi(2) \forall \tau \in X$. Let now κ, ϱ be such pairs of elements in X , for which there exists a sequence $n_1 < n_2 < \dots$ such that $\varphi(n_v) \rightarrow \kappa$, $\varphi(n_v + 1) \rightarrow \varrho$. Then $L(\varrho) \leftarrow \varphi(2n_v + 1) \rightarrow S(\kappa)$, and so $S(\kappa) = L(\varrho)$, $S(\kappa) = \kappa + \varphi(2)$, $L(\varrho) = \varrho + \varphi(2)$, i.e. $\kappa = \varrho$. Hence we can deduce easily that for each convergent sequence $\varphi(n_v)$ the sequence $\varphi(n_v + 1)$ converges as well, $\lim \varphi(n_v) = \kappa$ implies $\lim \varphi(n_v + 1) = \kappa$, and this gives almost immediately that $\varphi(n+1) - \varphi(n) \rightarrow 0$ ($n \rightarrow \infty$). $\square$

Let $Z = S[X_0]$, $W = L[X_0]$. It is clear that $g \in Z$, if there exists a suitable sequence $n_v \in \mathbb{N}$ such that $\varphi(2n_v - 1) \rightarrow h$, and $S(h) = g \leftarrow \varphi(2(2n_v - 1) + 1) = \varphi(4n_v - 1)$.

So $Z = \text{closure of } \{\varphi(4n - 1) | n \in \mathbb{N}\}$, and similarly $W = \text{closure of } \{\varphi(4n + 1) | n \in \mathbb{N}\}$.

Since the set $\{4n + 1, n = 0, 1, 2, \dots\}$ is a semigroup, $\{\varphi(4n + 1) | n = 0, 1, 2, \dots\}$ is a semigroup, and so W is a subgroup in X_0 . It is clear that

$$\varphi(4n - 1) + \varphi(4m + 1) \in Z,$$

for each n, m , consequently

$$Z + W \subseteq Z.$$

Similarly, $\varphi(4n - 1) + \varphi(4m - 1) \in W$, and so $Z + Z \subseteq W$. Since $X_0 = L[X_0] \cup S[X_0]$, $0 \in X_0$, there exists $h \in X_0$ such that $L(h) = 0$ or $S(h) = 0$. From Lemma 5 we get that $0 \in Z \cap W$. Then from the relations $Z + W \subseteq Z$, $Z + Z \subseteq W$ we have $Z = W = X_0$.

We have proved the following

Lemma 8. *We have*

$$L[X_0] = S[X_0] = X_0.$$

3. Let us consider now the special case $G = T$.

Assume that $\varphi \in \mathcal{A}_T^*(D[2, -1])$.

It is known that the only compact subgroups of T are T itself and the discrete groups $Z_m = \left\{ \frac{k}{m} \pmod{1}, k \in \mathbb{Z} \right\}$.

Let us assume first that $X = T$. If there exists an $h \in E_0$ for which the order of the element $h + \varphi(2)$ is infinite, then $\{k(h + \varphi(2)) | k \in \mathbb{N}\}$ is everywhere dense in T , and so by Lemma 6 we have $T \subseteq E_0$, i.e. $L[T] = S[T] = 0$, which leads to the trivial case $\varphi(n) = 0 \forall n$, $(n, 2) = 1$.

Assume now that E_0 contains infinitely many elements h_j each of which has finite order $o(h_j)$. Then the orders $o(h_j)$ cannot be bounded, and so by Lemma 6 we have immediately that E_0 is everywhere dense in T . From the continuity of L and S we get that $T = E_0$, i.e. that $\varphi(n) = 0 \forall n$, $(n, 2) = 1$.

There remains the case when $E_0 = \{h_1, h_2, \dots, h_r\}$. If $r = 1$, then Lemma 7 gives that $\varphi(n+1) - \varphi(n) \rightarrow 0$ ($n \rightarrow \infty$), and Wirsing's theorem implies that $\varphi(n) \equiv \lambda \log n \pmod{1}$. Assume now that $r \geq 2$. From $S(\tau - S(\tau)) = 0 \forall \tau \in T$, we get

that $\tau - S(\tau) \in E_0$. Let B_j be the set of those τ , for which $\tau - S(\tau) = h_j$. The sets B_j are disjoint closed sets, $\bigcup B_j = T$, furthermore $\tau - S(\tau)$ is a continuous function on the whole T . This is impossible if $r \geq 2$.

Let us assume now that $X = Z_M$ with a suitable M . If X_0 is the trivial group containing only the zero element, then $\varphi(2n+1) = 0$ for each $n \in \mathbb{N}$. Assume that X_0 contains at least two distinct elements. Since $L: X_0 \rightarrow X_0$, $S: X_0 \rightarrow X_0$ are such functions for which $L[X_0] = X_0$, $S[X_0] = X_0$, they are permutations in X_0 . Consequently there exists a unique $\gamma_1 \in X_0$, and a unique $\gamma_2 \in X_0$ such that $L(\gamma_1) = 0$, $S(\gamma_2) = 0$. Since $L(\gamma_1) = 0$ implies that $S(\gamma_1) = 0$, we have $\gamma_1 = \gamma_2 =: \gamma$. If $\tau \in X_0$, then $\tau - L(\tau)$, $\tau - S(\tau) \in X_0$, and so from (2.13), (2.14) we get

$$(3.1) \quad S(\tau) = L(\tau) = \tau - \gamma \quad \forall \tau \in X_0.$$

Hence, by Lemma 4 we obtain

$$(3.2) \quad S(\tau) = L(\tau) \quad \forall \tau \in X.$$

Since X is a discrete group the conditions $\varphi \in \mathcal{A}_T^*(D[2, -1])$ and (3.2) imply that

$$\varphi(2n-1) = L(\varphi(n)) = S(\varphi(n)) = \varphi(2n+1)$$

for each large n . But from this we get immediately that $\varphi(n) = 0$ for each odd n .

Collecting our results we get the following

Theorem. Let $\varphi \in \mathcal{A}_T^*(D[-2, 1])$. Then either

$$(1) \quad \varphi(n) = 0 \quad \text{for each odd } n,$$

or

$$(2) \quad \varphi(n) \equiv \lambda \log n \pmod{1} \quad (\forall n \in \mathbb{N}) \quad \text{with a suitable real number } \lambda.$$

Conversely, if $\varphi \in \mathcal{A}_T^*$ and satisfies (1) or (2), then $\varphi \in \mathcal{A}_T^*(D[-2, 1])$.

4. We are unable to prove a similar theorem for a general metrizable compact Abelian group.

The determination of $\mathcal{A}_G^*(A[-2, 1])$ can be solved easily. Let $\varphi \in \mathcal{A}_G^*(A[-2, 1])$. Then $\varphi(2n-1) - \varphi(n) \rightarrow c$, $\varphi(4n^2-1) - \varphi(2n^2) \rightarrow c$. Since $\varphi(4n^2-1) - \varphi(2n^2) = \varphi(2n-1) - \varphi(n) + (\varphi(2n+1) - \varphi(2n)) \rightarrow c$, we have $\varphi(2n+1) - \varphi(2n) \rightarrow 0$. Since $\varphi(2n+1) - \varphi(n+1) \rightarrow c$, the relations $\varphi(n+1) - \varphi(2n) \rightarrow -c$, $\varphi(n+1) - \varphi(n) \rightarrow \varphi(2) - c =: A$ also hold. Then

$$\varphi(n+1) - \varphi(n) = (\varphi(2n+2) - \varphi(2n+1)) + (\varphi(2n+1) - \varphi(2n)),$$

whence we get $A = 2A$, i.e. $A = 0$. So we have $\varphi \in \mathcal{A}_G^*(A[0, 1])$.

5. Now we formulate some conjectures.

Conjecture 1. Let $\varphi \in \mathcal{A}_T^*$ and assume that the set $\{\varphi(n) | n \in \mathbb{N}\}$ is everywhere dense in T . Let $\eta_n := (\varphi(n), \varphi(n+1), \dots, \varphi(n+k))$, and assume that the sequence $\{\eta_n | n \in \mathbb{N}\}$ is not everywhere dense in $T_{k+1} = T \times \dots \times T$. Then $\varphi(n) \equiv \lambda \log n \pmod{1}$ with a suitable $\lambda \in \mathbb{R}$.

Conjecture 2. Let $\varphi, \Psi \in \mathcal{A}_T^*$, and assume that both of the sets $\{\varphi(n) | n \in \mathbb{N}\}$, $\{\Psi(n) | n \in \mathbb{N}\}$ are everywhere dense in T . Assume furthermore that the sequence $\xi_n = (\varphi(n), \Psi(n+1))$ is not everywhere dense in $T_2 = T \times T$. Then there exist a suitable $\lambda \in \mathbb{R}$ and a rational s such that $\Psi(n) = s\varphi(n)$, $\varphi(n) \equiv \lambda \log n \quad \forall n \in \mathbb{N}$.

Conjecture 3. Let G_1, G_2 be metrically compact Abelian groups, $\varphi \in \mathcal{A}_{G_1}^*$, $\Psi \in \mathcal{A}_{G_2}^*$. Assume that the sets $\{\varphi(n) | n \in \mathbb{N}\}$, $\{\Psi(n) | n \in \mathbb{N}\}$ are everywhere dense in G_1, G_2 , respectively. Assume furthermore that the sequence $\theta_n = (\varphi(n), \Psi(n+1))$ is not everywhere dense in $H = G_1 \times G_2$. Then there exist integers P and Q such that

$$PG_1 = QG_2, \quad P\varphi(n) = Q\Psi(n),$$

and there exists a continuous homomorphism $\Lambda: R_x \rightarrow PG_1$ such that $\Lambda(n) = P\varphi(n) = Q\Psi(n)$.

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