Boundedness and asymptotic behavior of solutions of second order nonlinear differential equations

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

In this paper we study the asymptotic behavior of the solutions of a broad class of second order nonlinear differential equations, namely,

$$(a(t)x')' + h(t, x, x') + q(t)f(x) = e(t, x, x'). \tag{E}$$

Equation (E) can be interpreted as the equation of the motion of a mechanical system with one degree of freedom having kinetic energy $a(t)[x']^2/2$ and potential energy $q(t)\int_0^x f(u)\,du$. The system is under the action of non-potential forces $h(t, x, x')$ and $e(t, x, x')$. The force h may typically be damping while e denotes a perturbation.

After beginning with some continuability and boundedness results, we next give sufficient conditions for all solutions of (E) to converge to zero. This is followed by some results which give sufficient conditions for the convergence to zero of certain classes of solutions of (E). In so doing, we extend some recent results of Baker [1], Ballieu and Peiffer [2], Lazer [10], Mamii and Mirzov [11], Scott [12, 13], Vornicescu [15], Willett and Wong [16], Wong [17, 18] and the present authors [3—5, 7—9, 14].

2. Continuability and boundedness

Consider the equation

$$(a(t)x')' + h(t, x, x') + q(t)f(x) = e(t, x, x'), \tag{1}$$

where $a, q\colon [t_0, \infty)\to\mathbf{R}$; $f\colon \mathbf{R}\to\mathbf{R}$; $h, e\colon [t_0, \infty)\times\mathbf{R}^2\to\mathbf{R}$ are continuous, $a(t)>0$ and $q(t)>0$. We will write equation (1) as the system

$$x' = y,$$
$$y' = [-a'(t)y - h(t, x, y) - q(t)f(x) + e(t, x, y)]/a(t). \tag{2}$$

For any function Q we let $Q(t)_+=\max\{Q(t),0\}$ and $Q(t)_-=\max\{-Q(t),0\}$ so that $Q(t)=Q(t)_+-Q(t)_-$. We define $F(x)=\int_0^x f(u)\,du$ and assume that

(3) $$xf(x)>0 \quad \text{for} \quad x\neq 0;$$

there are nonnegative continuous functions r_0, r_1, r_2: $[t_0,\infty)\to R$ and a number m $(0\leqq m\leqq 1)$ such that

(4) $$|e(t,x,y)|\leqq r_0(t)+r_1(t)F^{1/2}(x)+r_2(t)|y|^m;$$

and there exists a continuous function w: $[t_0,\infty)\to R$ such that

(5) $$0\leqq w(t)y^2\leqq yh(t,x,y).$$

We will make use of the following additional assumptions on the coefficient functions in equation (1):

(6) $$\int_0^\infty [(a(s)q(s))'/a(s)q(s)+2w(s)/a(s)]_-\,ds<\infty;$$

(7) $$F(x)\to\infty \quad as \quad |x|\to\infty;$$

(8) $$\int_0^\infty [r_0(s)+r_1(s)+r_2(s)(q(s)/a(s))^{m/2}]/(a(s)q(s))^{1/2}<\infty.$$

The proofs of our theorems are based upon Lyapunov's direct method. For a given solution $(x(t),y(t))$ of (2), we define the Lyapunov function

$$V(t)=a(t)y^2(t)/q(t)+2F(x(t)).$$

Then

(9) $$V'(t)=2a(t)y(t)y'(t)/q(t)+y^2(t)(a(t)/q(t))'+2y(t)f(x(t))=$$
$$=-(a(t)q(t))'y^2(t)/q^2(t)-2y(t)h(t,x(t),y(t))/q(t)+2y(t)e(t,x(t),y(t))/q(t).$$

We begin with a continuability result that extends similar results in [4] and [14].

Theorem 1. *If* (3)—(5) *hold, then all solutions of* (2) *can be defined for all* $t\geqq t_0$.

Proof. Suppose there is a solution $(x(t),y(t))$ of (2) such that $\lim_{t\to T^-}[|x(t)|+ + y(t)|]=\infty$. Then from (9) we obtain

$$V'(t)\leqq (a(t)y^2(t)/q(t))[(a(t)q(t))'/a(t)q(t)+2w(t)/a(t)]_-+$$
$$+2\{[r_0(t)+r_2(t)(q(t)/a(t))^{m/2}]/(a(t)q(t))^{1/2}\}[a(t)y^2(t)/q(t)+1]+$$
$$+2r_1(t)[F(x(t))+a(t)y^2(t)/q(t)]/(a(t)q(t))^{1/2}.$$

So

(10) $$V'(t)\leqq [(a(t)q(t))'/a(t)q(t)+2w(t)/a(t)]_-V(t)+$$
$$+[2/(a(t)q(t))^{1/2}][r_0(t)+r_1(t)+r_2(t)(q(t)/a(t))^{m/2}][V(t)+1].$$

Integrating the last inequality and then applying Gronwall's inequality we have that $V(t)$ is bounded on $[t_0,T)$. Hence there exist positive constants k_1 and k_2

such that $y^2(t) \leqq k_1 q(t)/a(t) \leqq k_2$ on $[t_0, T)$. This implies that $y(t)=x'(t)$ is bounded on $[t_0, T)$ and an integration shows that $x(t)$ is also bounded on $[t_0, T)$ contradicting the assumption that $(x(t), y(t))$ is a solution of (2) with finite escape time.

The following theorem gives sufficient conditions for all solutions of (1) to be bounded.

Theorem 2. *If conditions* (3)—(8) *hold, then for every solution* $x(t)$ *of* (1) *the function* $V(t)$ *has a finite limit as* $t \to \infty$ *and, consequently, the solution* $x(t)$ *and the function* $a(t)(x'(t))^2/q(t)$ *are bounded.*

PROOF. Let $x(t)$ be a solution of (1). It follows from (6), (8), and (10) that

$$(V(t)+1)' \leqq P_1(t)(V(t)+1)$$

where $P_1(t) \geqq 0$ and $\int_{t_0}^{\infty} P_1(s)\,ds < \infty$. Applying Gronwall's inequality we see that $V(t)$ is bounded, say $V(t) \leqq k$ for $t \geqq t_0$. Notice that from (10) we also have

$$V'(t) \leqq P_1(t)(V(t)+1) \leqq P_1(t)(k+1).$$

The last inequality implies that $V'(t)_+ \leqq P_1(t)(k+1)$, so

$$\int_{t_0}^{t} V'(t)_+\,ds \leqq (k+1) \int_{t_0}^{t} P_1(s)\,ds.$$

Since $V'(t)_+ = V'(t) + V'(t)_-$,

$$\int_{t_0}^{t} V'(t)_+\,ds = V(t) - V(t_0) + \int_{t_0}^{t} V'(t)_-\,ds$$

from which it follows that

$$\int_{t_0}^{t} V'(t)_-\,ds = V(t_0) + \int_{t_0}^{t} V'(t)_+\,ds - V(t) \leqq V(t_0) + \int_{t_0}^{t} V'(t)_+\,ds.$$

Thus we have

$$\int_{t_0}^{t} |V'(s)|\,ds = \int_{t_0}^{t} [V'(t)_+ + V'(t)_-]\,ds \leqq V(t_0) + 2\int_{t_0}^{t} V'(t)_+\,ds \leqq$$

$$\leqq V(t_0) + 2(k+1) \int_{t_0}^{\infty} P_1(s)\,ds$$

and so V is of bounded variation which completes the proof.

Remark. If $m=1$ in (4), it is easy to see that (6) and (8) could be replaced by

$$(6') \qquad \int_{t_0}^{\infty} [(a(s)q(s))'/a(s)q(s) + 2w(s)/a(s) - 2r_2(s)/a(s)]_-\,ds < \infty$$

and

$$\int_{t_0}^{\infty} \{[r_0(s)+r_1(s)]/(a(s)q(s))^{1/2}\}\, ds < \infty \tag{8'}$$

respectively.

Remark. Applications of Schwarz's inequality reveal that condition (8) is less restrictive than other types of conditions that can and have been imposed on the functions r_0, r_1, and r_2. For example, it is easy to see that if

$$\int_{t_0}^{\infty} [r_0(s)/q^{1/2}(s)]\, ds < \infty \tag{P$_1$}$$

and

$$\int_{t_0}^{\infty} [r_0(s)/a(s)q^{1/2}(s)\, ds < \infty \tag{P$_2$}$$

hold, then

$$\Big\{\int_{t_0}^{\infty} [r_0(s)/(a(s)q(s))^{1/2}]\, ds\Big\}^2 = \Big\{\int_{t_0}^{\infty} [r_0^{1/2}(s)/a^{1/2}(s)q^{1/4}(s)][r_0^{1/4}(s)/q^{1/4}(s)]\, ds\Big\}^2 \leqq$$
$$\leqq \Big\{\int_{t_0}^{\infty} [r(s)/a(s)q^{1/2}(s)]\, ds\Big\}\Big\{\int_{t_0}^{\infty} [r_0(s)/q^{1/2}(s)]\, ds\Big\} < \infty$$

So (8) holds. Conditions such as (P_1) and (P_2) above arise in reconstructing V from V' in a manner different from that used in the proof of Theorem 1. Here, for example, we could proceed as follows:

$$r_0(t)y/q(t) = [r_0(t)/q^{1/2}(t)][y/q^{1/2}(t)] \leqq [r_0(t)/q^{1/2}(t)][y^2/q(t)+1] \leqq$$
$$\leqq [r_0(t)/a(t)q^{1/2}(t)]V + r_0(t)/q^{1/2}(t).$$

The authors have examined sixty-four different combinations of conditions on r_0, r_1 and r_2 resulting from various forms of constructions of this type, and all these combinations of conditions imply condition (8). Consequently, Theorem 2 extends a number of known boundedness results such as those in [1, 4, 7, 8, 11, 14, 16] and special cases of those in [3, 5].

3. Convergence to zero

With continuability and boundedness criteria established, we are now ready to impose additional conditions which will ensure that solutions of (1) tend to zero as $t\to\infty$. For this purpose we establish the following result.

Proposition 3. *For any given positive constants* K_1 *and* ε *there exists a positive constant* A *such that* $A<\sup\{2F(x)/xf(x)\colon x\neq 0\}$ *and*

$$2F(x) - Axf(x) < \varepsilon \tag{$*$}$$

for all x *such that* $|x| \leqq K_1$.

PROOF. Let $K_1>0$ and $\varepsilon>0$ be given and let $B=\sup\{2F(x)/xf(x)\colon x\neq 0\}$. If $B=\infty$, then by a known result (see Karsai [9] or Scott [13]) there is a constant $A>0$ such that $(*)$ holds and the proof is complete for this case. If $B<\infty$, first observe that there exists $K_2>0$ such that $|f(x)|<K_2$ for all x satisfying $|x|<K_1$. Choose $\varepsilon_1>0$ such that $\varepsilon_1<\min\{B, \varepsilon/2K_1K_2\}$. Then for $x\neq 0$ we have

$$2F(x)/xf(x) < (B-\varepsilon_1)+2\varepsilon_1,$$

so

$$2F(x)-(B-\varepsilon_1)xf(x) < 2\varepsilon_1 xf(x) < \varepsilon$$

for $|x|<K_1$ and we see that $(*)$ holds with $A=B-\varepsilon_1<B$. Moreover, if $x=0$, then clearly $(*)$ holds with $A=B-\varepsilon_1$ since $F(0)=f(0)=0$. This completes the proof of the proposition.

Theorem 4. *Suppose that in addition to* (3)—(8) *there exist nonnegative continuous functions* $w_1, \alpha\colon [t_0, \infty)\to R$ *such that for all bounded* x

$$|h(t,x,y)| \leqq |y|\,w_1(t), \tag{11}$$

$$I(t) = \int_{t_0}^{t} \alpha(s)\,ds \to \infty \quad as \quad t\to\infty, \tag{12}$$

there is a number σ $(0<\sigma\leqq 1)$ *such that* $0<\lambda<\sup\{2F(x)/xf(x)\colon x\neq 0\}$ *implies*

$$\limsup_{t\to\infty}\,(1/I(t))\int_{t_0}^{t}\big\{\big[(a(s)q(s))'/a(s)q(s)+2w(s)/a(s)\big]I(s)-(1+\lambda)\alpha(s)\big\}_-\,ds = \tag{13}$$
$$= l(\lambda) < 1-\sigma,$$

$$\int_{t_0}^{t} \big|(\alpha(s)/q(s))'\big|\big(a(s)q(s)\big)^{1/2} = o\big(I(t)\big), \quad t\to\infty, \tag{14}$$

$$\alpha(t)\big(a(t)/q(t)\big)^{1/2} = o\big(I(t)\big), \quad t\to\infty, \tag{15}$$

and

$$\int_{t_0}^{t} \big[w_1(s)\alpha(s)/\big(a(s)q(s)\big)^{1/2}\big]\,ds = o\big(I(t)\big), \quad t\to\infty. \tag{16}$$

Then every solution $x(t)$ *of* (1) *satisfies* $x(t)\to 0$ *as* $t\to\infty$.

PROOF. Let $x(t)$ be a solution of (1) and let $V(t)$ be defined as before. Then from Theorem 2 we have that $x(t)$ is bounded and $V(t)$ has a finite limit as $t\to\infty$. Suppose that $V(t)\to\lambda_1>0$ as $t\to\infty$. If ε is any given positive number, then by Proposition 3 there exists a constant A such that $0<A<\sup\{2F(x)/xf(x)\colon x\neq 0\}$ and

$$2F\big(x(t)\big)-Ax(t)f\big(x(t)\big) < \varepsilon \tag{17}$$

for $t\geqq t_0$. By observing that $\big(a(t)x(t)y(t)\big)'=x(t)\big(a(t)y(t)\big)'+a(t)y^2(t)$, $V(t)$ can be rewritten in the form

$$V(t) = (A+1)a(t)y^2(t)/q(t)-A\big(a(t)x(t)y(t)\big)'/q(t)-Ax(t)f\big(x(t)\big)+$$
$$+2F\big(x(t)\big)+Ax(t)\big[e\big(t,x(t),y(t)\big)-h\big(t,x(t),y(t)\big)\big]/q(t).$$

Let K and L denote positive constants satisfying $|x(t)|\leqq K$ and $V(t)\leqq L^2$; then $|y(t)|\big(a(t)/q(t)\big)^{1/2}\leqq V^{1/2}(t)\leqq L$. Therefore, from (17) we have

$$(18)\quad V(t) \leqq (A+1)\,a(t)\,y^2(t)/q(t)-A\big(a(t)\,x(t)\,y(t)\big)'/q(t)+\varepsilon+AKr_0(t)/q(t)+$$

$$+ALKr_1(t)/q(t)+AL^mKr_2(t)\big(q(t)/a(t)\big)^{m/2}/q(t)+AKLw_1(t)/\big(a(t)\,q(t)\big)^{1/2}.$$

Now define $H(t)=V(t)I(t)$; then $H'(t)=V'(t)I(t)+V(t)\alpha(t)$. From (9), (18), and the fact that $\alpha(t)\geqq 0$ we have

$$H'(t) \leqq (1+A)\,a(t)\,y^2(t)\alpha(t)/q(t)-A\alpha(t)\big(a(t)\,x(t)\,y(t)\big)'/q(t)+$$

$$+AKLw_1(t)\alpha(t)/\big(a(t)\,q(t)\big)^{1/2}+\varepsilon\alpha(t)$$

$$+KA\alpha(t)\big[r_0(t)+Lr_1(t)+L^m\,r_2(t)\big(q(t)/a(t)\big)^{m/2}\big]/q(t)-$$

$$-\big[\big(a(t)\,q(t)\big)'/a(t)\,q(t)+2w(t)/a(t)\big]\big[a(t)\,y^2(t)\,I(t)/q(t)\big]+$$

$$+2L\big[r_0(t)+Lr_1(t)+L^m\,r_2(t)\big(q(t)/a(t)\big)^{m/2}\big]\,I(t)/\big(a(t)\,q(t)\big)^{1/2}.$$

Choose $T>t_0$ such that if $t\geqq T$, then $\int\limits_T^t \big[r_i(s)/\big(a(s)\,q(s)\big)^{1/2}\big]\,ds<\varepsilon$ for $i=0, 1,$ $|V(t)-\lambda_1|<\varepsilon$, and $\int\limits_T^t \big\{r_2(s)\big(q(s)/a(s)\big)^{m/2}/\big(a(s)\,q(s)\big)^{1/2}\big\}\,ds<\varepsilon$. Then integrating H' we have

$$H(t) \leqq H(T)-A\int\limits_T^t \big[\alpha(s)\big(a(s)\,x(s)\,y(s)\big)'/q(s)\big]\,ds+\varepsilon\int\limits_T^t \alpha(s)\,ds+$$

$$+AKL\int\limits_T^t \big[w_1(s)\,\alpha(s)/\big(a(s)\,q(s)\big)^{1/2}\big]\,ds$$

$$+AK\int\limits_T^t \big\{\alpha(s)\big[r_0(s)+Lr_1(s)+L^m\,r_2(s)\big(q(s)/a(s)\big)^{m/2}\big]/q(s)\big\}\,ds+$$

$$+2L\int\limits_T^t \big\{\big[r_0(s)+Lr_1(s)+L^m\,r_2(s)\big(q(s)/a(s)\big)^{1/2}\big]I(s)/\big(a(s)\,q(s)\big)^{1/2}\big\}\,ds+$$

$$+\int\limits_T^t \big\{I(s)\big[\big(a(s)\,q(s)\big)'/a(s)\,q(s)+2w(s)/a(s)\big]-(1+A)\,\alpha(s)\big\}_-\,V(s)\,ds.$$

Integrating the first integral in the last inequality by parts and noticing that $I(s)=$

$= \int_{t_0}^{s} \alpha(u)\,du = I(T) + \int_{T}^{t} \alpha(s)\,ds$, we obtain from the choice of T that

$$H(t) \leqq C_1 - A\alpha(t)a(t)x(t)y(t)/q(t) + A\int_T^t \big[(\alpha(s)/q(s))' a(s)x(s)y(s)\big]\,ds + \tag{19}$$

$$+AKL\int_T^t \big[w_1(s)\alpha(s)/(a(s)q(s))^{1/2}\big]\,ds + Q(t)(\lambda_1+\varepsilon) +$$

$$+AK\int_T^t \big\{\alpha(s)\big[r_0(s)+Lr_1(s)+L^m r_2(s)\big(q(s)/a(s)\big)^{m/2}\big]/q(s)\big\}\,ds +$$

$$+\varepsilon\int_T^t \alpha(s)\,ds + (1+L+L^m)\,\varepsilon\int_T^t \alpha(s)\,ds$$

for some positive constant C_1 and

$$Q(t) = \int_T^t \big\{I(s)\big[(a(s)q(s))'/a(s)q(s) + 2w(s)/a(s)\big] - (1+A)\alpha(s)\big\}_-\,ds.$$

Using the fact that $|x(t)| \leqq K$ and $(a(t)/q(t))^{1/2}|y(t)| \leqq L$ we obtain the estimate

$$H(t) \leqq C_1 + AKL\alpha(t)\big(a(t)/q(t)\big)^{1/2} + AKL\int_T^t \big|(\alpha(s)/q(s))'\big|\big(a(s)q(s)\big)^{1/2}\,ds +$$

$$+AKL\int_T^t \big[w_1(s)\alpha(s)/(a(s)q(s))^{1/2}\big]\,ds + Q(t)(\lambda_1+\varepsilon) +$$

$$+[AK(1+L+L^m)\varepsilon \sup\big\{\alpha(s)\big((a(s)/q(s))^{1/2}\colon T \leqq s \leqq t\big\} +$$

$$+[1+2L(1+L+L^m)]\,\varepsilon\int_T^t \alpha(s)\,ds.$$

Dividing the last inequality by $I(t)$ and letting $t\to\infty$ we have from (12)—(16) that

$$\lambda < [1+2L(1+L+L^m)]\varepsilon + (1-\sigma)(\lambda_1+\varepsilon) + \varepsilon$$

which yields a contradiction for $\varepsilon > 0$ sufficiently small. The contradiction is derived from the assumption that $\lambda_1 > 0$, so theorem is proved.

Remark. Theorem 4 improves Theorem 1 in [9] even in the case $h \equiv 0$. Specifically, condition (13) is weaker than the corresponding one in [9]: for every $\lambda > 0$

$$\int_{t_0}^{\infty} \big[(a(s)q(s))'/a(s)q(s) - (\lambda+1)\alpha(s)/I(s)\big]_-\,ds < \infty. \tag{20}$$

To verify this, we first prove that (20) implies (13). Suppose that (20) holds and for a given $\varepsilon > 0$ let T be so large that

$$\int_T^{\infty} \big[(a(s)q(s))'/a(s)q(s) - (\lambda+1)\alpha(s)/I(s)\big]_-\,ds < \varepsilon.$$

Then

$$\limsup_{t\to\infty}[1/I(t)]\int_{t_0}^{t}\left[\left(a(s)q(s)\right)'I(s)/a(s)q(s)-(\lambda+1)\alpha(s)\right]_-\,ds<$$

$$<\int_{T}^{\infty}\left[\left(a(s)q(s)\right)'/a(s)q(s)-(\lambda+1)\alpha(s)/I(s)\right]_-<\varepsilon;$$

therefore, (13) holds.

On the other hand, we can show an equation for which condition (20) is not satisfied, but (13) is. Consider the equation

$$x''+q(t)x=0,\quad t\geqq t_0=0 \tag{21}$$

where the coefficient q is defined as follows:

$$q(t)=\begin{cases}e^{t-n/3} & \text{if}\quad n\leqq t\leqq n+2/3\\ e^{2(n+1)/3} & \text{if}\quad n+2/3<t<n+1\end{cases}$$

for $n=0,1,2,\dots$. Let $\alpha(t)=1/(1+t)$ so that $I(t)=\ln(1+t)$.

Notice that

$$q'(t)/q(t)-(1+\lambda)\alpha(t)/I(t)=\begin{cases}1-(1+\lambda)/(t+1)\ln(t+1), & n<t<n+2/3\\ -(1+\lambda)/(t+1)\ln(t+1), & n+2/3\leqq t\leqq n+1,\end{cases}$$

so

$$\int_0^\infty[q'(s)/q(s)-(1+\lambda)\alpha(s)/I(s)]_-\,ds\geqq(1+\lambda)\sum_{i=0}^{\infty}\int_{i+2/3}^{i+1}[1/(s+1)\ln(s+1)]\,ds\geqq$$

$$\geqq(1+\lambda)\sum_{i=0}^{\infty}[1/\ln(i+2)]\int_{i+2/3}^{i+1}[(1/(s+1)]\,ds=$$

$$=(1+\lambda)\sum_{i=1}^{\infty}[1/\ln(i+2)]\ln[(i+2)/(i+5/3)].$$

It is easy to see by L'Hospital's rule that $(i+2)\ln(i+2)/[(i+5/3]\to 1/3$ as $i\to\infty$. Thus there exist $N>1$ and $0<C<1/3$ such that

$$[1/\ln(i+2)]\ln[(i+2)/(i+5/3)]>C/(i+2)\ln(i+2),\quad i\geqq N.$$

Hence,

$$\int_0^\infty[q'(s)/q(s)-(1+\lambda)\alpha(s)/I(s)]_-\,ds\geqq(1+\lambda)C\sum_{i=N}^{\infty}[1/(i+2)\ln(i+2)].$$

Since the series on the right diverges to $+\infty$, we see that (20) does not hold. To see that (13) holds, let

$$K(t)=[1/I(t)]\int_0^t[q'(s)I(s)/q(s)-(1+\lambda)\alpha(s)]_-\,ds;$$

then

$$K(n+1)=\sum_{i=0}^{n}\int_{i}^{i+2/3}[\ln(s+1)-(1+\lambda)/(s+1)]_-\,ds+\sum_{i=0}^{N}\int_{i+2/3}^{i+1}[(i+\lambda)/(s+1)]\,ds.$$

Notice that

$$\int_{i+2/3}^{i+1} [(1+\lambda)/(s+1)]\, ds \leqq [(1+\lambda)/(i+5/3)] \int_{i+2/3}^{i+1} ds = (1+\lambda)/3(i+5/3),$$

so

$$I(n+1)K(n+1) \leqq \sum_{i=0}^{n} \int_{i}^{i+2/3} [\ln(s+1)-(1+\lambda)/(s+1)]_{-}\, ds + [(1+\lambda)/3] \sum_{i=0}^{n} (i+5/3)^{-1}.$$

Thus we have

$$K(n+1) \leqq [1/\ln(n+2)] \times$$
$$\times \Big\{ \sum_{i=0}^{n} \int_{i}^{i+2/3} [\ln(s+1)-(1+\lambda)/(s+1)]_{-}\, ds + [(1+\lambda)/3] \sum_{i=0}^{n} (i+5/3)^{-1} \Big\}$$

Now let k be a positive integer such that $\ln(s+1)-(1+\lambda)/(s+1) \geqq 0$ got $s \geqq k$. Then

$$K(n+1) \leqq [1/\ln(n+2)] \times$$
$$\times \Big\{ \sum_{i=0}^{k} \int_{i}^{i+2/3} [\ln(s+1)-(1+\lambda)/(s+1)]_{-}\, ds + [(1+\lambda)/3\ln(n+2)] \sum_{i=0}^{n} (i+5/3)^{-1} \Big\} \leqq$$
$$\leqq [1/\ln(n+2)] \Big\{ \sum_{i=0}^{k} \int_{i}^{i+2/3} [\ln(s+1)-(1+\lambda)/(s+1)]_{-}\, ds + [(\lambda+1)/3\ln(n+2)] \sum_{i=1}^{n+1} (1/i).$$

Now

$$\limsup_{n\to\infty} [1/\ln(n+2)] \Big\{ \sum_{i=0}^{k} \int_{i}^{i+2/3} [\ln(s+1)-(1+\lambda)/(s+1)]_{-}\, ds \Big\} = 0,$$

hence

$$\limsup_{n\to\infty} K(n+1) \leqq \limsup_{n\to\infty} [(\lambda+1)/3\ln(n+2)] \sum_{i=1}^{n+1} (1/i) = (\lambda+1)/3.$$

For equation (21) we have $\sup\{2F(x)/xf(x)\colon x\neq 0\}=1$. If $0<\lambda<1$ then $(1+\lambda)/3<1$, so condition (13) is satisfied for equation (21).

Example. Consider the motion of a pendulum whose length at time t is given by $l(t)$ where $l'(t)\geqq 0$. Assume that viscous fiiction acts on the pendulum in such a way that the damping force is proportional to its velocity. Let the position of the pendulum in the plane be described by its length $l(t)$ and the angle x between the axis directed vertically downward and the pendulum. Then the kinetic energy and the resultant of the forces are

$$T(t) = (m/2)\big(l^2(t)[x'(t)]^2+[l'(t)]^2\big)$$

and

$$Q(t) = -mgl(t)\sin x(t) - h(t)x'(t),$$

where m is the mass of the pendulum material, g denotes the gravitational constant and $h(t)\geqq 0$ is the frictional coefficient at the moment t. The Lagrange's equation of the second kind for this motion is as follows:

$$\big(ml^2(t)x'\big)' + h(t)x' + mgl(t)\sin x = 0, \quad -\pi \leqq x < \pi. \tag{22}$$

Consider the Lyapunov function

$$V(t) = l(t)[x'(t)]^2/g + 2(1-\cos x).$$

By (9) we have $V'(t) \leqq 0$. Therefore, if the initial values $x(0)$, $x'(0)$ are small enough, then $|x(t)| < \pi/2$ for all $t \geqq 0$, so condition (7) is not needed here.

It is well-known [2] that a large damping term can destroy the asymptotic stability with respect to x. We will now consider the following situation. Let the length of the arm change by the law $l(t) = (t+1)^k$, where $0 \leqq k < 2$ is fixed. Under how large a damping will $x(t)$ still tend to 0 as $t \to \infty$?

Choose $\alpha(t) = (t+1)^r$ $(-1 < r < 2)$; then $I(t) = [(t+1)^{r+1} - 1]/(r+1) \to \infty$ as $t \to \infty$. Simple computations show that conditions (14) and (15) are satisfied for all r. To check condition (13) consider the expression

$$[3l'(t)/l(t) + 2h(t)/ml^2(t)]\, I(t) - (1+\lambda)\alpha(t) \geqq$$

$$\geqq [3k/(r+1) - (1+\lambda)](t+1)^r - 3k/(r+1)(t+1).$$

Clearly we can choose r close enough to $t_0 - 1$ so that

$$3k/(r+1) - 1 - \sup\{2(1-\cos x)/x \sin x \colon 0 < |x| \leqq \pi/2\} > 0$$

holds, so (13) holds.

If $\int_0^\infty [h(s)/(s+1)^{3k/2-r}]\, ds < \infty$; then (16) is satisfied, while if this integral diverges, then by L'Hospital's rule we have

$$\lim_{t\to\infty} (t+1)^{-r-1} \int_0^t [h(s)/(s+1)^{3k/2-r}]\, ds = \lim_{t\to\infty} h(t)(t+1)^{-3k/2},$$

so we have proved the following result.

Proposition. *Suppose that the length of the arm in* (22) *changes by the law* $l(t) = (t+1)^k$ $(0 \leqq k < 2)$. *If the frictional coefficient* $h(t)$ *satisfies the relation* $h(t) = o((t+1)^{3k/2})$ *as* $t \to \infty$, *then for every motion with sufficiently small initial values* $x(0)$, $x'(0)$ *the angle of deviation* $x(t)$ *tends to* 0 *as* $t \to \infty$.

If the functions α and q belong to $C^2[t_0, \infty)$ then the proof of Theorem 4 can be modified to obtain:

Theorem 5. *Suppose that all the assumptions of Theorem* 4 *are satisfied except* (14). *If, in addition*

$$a(t)\left|(\alpha(t)/q(t))'\right| = o(I(t)), \quad t \to \infty, \tag{14'}$$

and

$$\int_{t_0}^t [(a(s)(\alpha(s)/q(s))')']_- = o(I(t)), \quad t \to \infty, \tag{14''}$$

then every solution $x(t)$ *of* (1) *satisfies* $x(t) \to 0$ *as* $t \to \infty$.

PROOF. Let $x(t)$ be a solution of (1). Proceed exactly as in the proof of Theorem 4 until inequality (19) is obtained. Then integrate the first integral in the right number of (19) by parts to obtain

$$(23)\qquad A\int_T^t (\alpha(s)/q(s))' a(s)x(s)y(s)\,ds = C_2+(A/2)(\alpha(t)/q(t))' a(t)x^2(t)-$$
$$-(A/2)\int_T^t [a(s)(\alpha(s)/q(s))']' x^2(s)\,ds$$

where C_2 is a constant. The remainder of the proof is as before using (14′) and (14″) in place of (14).

Remark. Theorem 5 extends a similar result obtained in [9] and Theorems 4 and 5 generalize similar results of BALLIEU and PEIFFER [2], LAZER [10], and SCOTT [12, 13]. Theorem 4 also improves Theorem 8 in [4] and Theorem 1.1 in [16].

In what follows it will be convenient to classify the solutions of (1) as follows. A solution $x(t)$ of (1) will be called nonoscillatory if there exists $t_1 \geqq t_0$ such that $x(t)\neq 0$ for $t\geqq t_1$; the solution will be called oscillatory if for any $t_1\geqq t_0$ there exist t_2 and t_3 satisfying $t_1<t_2<t_3$ and $x(t_2)x(t_3)<0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is eventually nonnegative or nonpositive. With this classification we have the following two corollaries.

Corollary 6. *If all the conditions of either Theorem* 4 *or Theorem* 5 *hold with condition* (15) *replaced by*

$$(15')\qquad \int_{t_0}^{\infty} \{\alpha(s)[r_0(s)+r_1(s)+r_2(s)(q(s)/a(s))^{m/2}]/q(s)\}\,ds<\infty,$$

then every not eventually monotonic solution $x(t)$ of (1) *satisfies $x(t)\to 0$ as $t\to\infty$.*

Corollary 7. *If* (15′) *holds and all other hypotheses of Theorem* 5 *hold except* (14′) *and* (15), *then every oscillatory or Z-type solution $x(t)$ of* (1) *satisfies $x(t)\to 0$ as $t\to\infty$.*

The proofs of both corollaries consist, in view of the fact that $V(t)$ has a finite limit as $T\to\infty$, of showing that there exists a sequence $\{t_n\}$ for which $t_n\to\infty$ and $V(t_n)\to 0$ at $n\to\infty$. For Corollary 6, choose $\{t_n\}$ so that $x'(t_n)=0$; then $-A\alpha(t_n)a(t_n)x(t_n)y(t_n)/q(t_n)=0$ in (19) for each n and (15) is not needed. For Corollary 7, choose $\{t_n\}$ so that $x(t_n)=0$; then in (23) $(A/2)(\alpha(t_n)/q(t_n))' a(t_n)x^2(t_n)=0$ for all n and (14′) is not needed.

Remark. Theorem 7 in [4], Theorem 3.1 in [8], Theorem 4 in [14], and the Theorem in [15] all obtain the conclusion of Corollary 7 for special cases of (1). Corollary 7 improves these results not only in the sense that it applies to a more general equation, but also in that all of the other results require $(a(t)q(t))'\geqq 0$ for all $t\geqq t_0$ and conditions that imply (13). Corollary 7 also extends similar results obtained by Wong in [17, 18] for special cases of (1).

It is suspected from the results on special cases of (1) that the convergence to zero of the nonoscillatory solutions of (1) can be obtained with much less restrictive conditions than (12)—(16). For example, in [8] it was proved that all the nonoscillatory solutions of equation (1) with $h\equiv e\equiv 0$ tend to zero as $t\to\infty$ if and only if

$$\int_0^\infty (1/a(t))\int_0^t q(s)\,ds\,dt=\infty. \tag{24}$$

This result can be extended to equation (1) as follows.

Theorem 8. *Suppose that* (3)—(8) *and* (11) *hold. If*

$$\int_{t_0}^\infty (1/a(s))\int_{t_0}^s [r_0(u)+r_1(u)+r_2(u)\big(q(u)/a(u)\big)^{m/2}+ \tag{25}$$
$$+w_1(u)\big(q(u)/a(u)\big)^{1/2}-K_3q(u)]\,du\,ds+\int_{t_0}^\infty [1/a(s)]\,ds=-\infty$$

for any positive constant K_3, *then every nonoscillatory or Z-type solution* $x(t)$ *of* (1) *satisfies* $x(t)\to 0$ *as* $t\to\infty$.

PROOF. Let $x(t)$ be a nonoscillatory or Z-type solution of (1), say $x(t)\geqq 0$ for $t\geqq t_1\geqq t_0$; then as in the proof of Theorem 4 there is a constant $L>0$ such that $V(t)\leqq L^2$. We first show that $\liminf_{t\to\infty} x(t)=0$.

If this is not the case, then since $x(t)$ is bounded from above and from below there exist $t_2\geqq t_1$ and $\delta>0$ such that $f(x(t))\geqq\delta>0$ for $t\geqq t_2$. Now from (1), (4), and (11)

$$\big(a(t)y(t)\big)'+q(t)f\big(x(t)\big)\leqq r_0(t)+r_1(t)F^{1/2}\big(x(t)\big)+r_2(t)|y(t)|^m+w_1(t)|y(t)|,$$

so

$$\big(a(t)y(t)\big)'\leqq r_0(t)+Lr_1(t)+L^m r_2(t)\big(q(t)/a(t)\big)^{m/2}+Lw_1(t)\big(q(t)/a(t)\big)^{1/2}-\delta q(t) \tag{26}$$

and integrating twice we obtain

$$x(t)\leqq x(t_2)+a(t_2)|y(t_2)|\int_{t_2}^t [1/a(s)]\,ds+$$
$$+\int_{t_2}^t [1/a(s)]\int_{t_2}^s [r_0(u)+Lr_1(u)+L^m r_2(u)\big(q(u)/a(u)\big)^{m/2}+$$
$$+Lw_1(u)\big(q(u)/a(u)\big)^{1/2}-\delta q(u)]\,du\,ds.$$

Hence, according to (25), $x(t)\to-\infty$ as $t\to\infty$ which is a contradiction.

If $x(t)$ is monotonic on an interval $[T,\infty)$, then the proof is complete. If $x(t)$ is not monotonic on any interval $[T,\infty)$, then let $\{\tau_n\}$ be a sequence of minima of $x(t)$ such that $\tau_n\to\infty$ as $n\to\infty$.

Since $x'(\tau_n)=0$ $(n=1,2,\ldots)$ and $x(\tau_n)\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty} V(\tau_n)=\lim_{n\to\infty} F\big(x(\tau_n)\big)=0.$$

On the other hand, by Theorem 2, $V(t)$ has a finite limit as $t\to\infty$. Therefore, $V(t)\to 0$ and, consequently, $x(t)\to 0$ as $t\to\infty$.

Remark. Notice that condition (25) implies condition (24) which, as noted above, is a necessary and sufficient condition for the conclusion of Theorem 8 when $h\equiv e\equiv 0$. Condition (24) is also necessary in Theorem 8. Namely, it can be proved (see [4], Theorem 6) that if all the conditions of Theorem 8 except (25) hold, $\int_{t_0}^{\infty}[1/a(t)]\int_{t_0}^{t}q(s)\,ds\,dt<\infty$, and

$$\int_{t_0}^{\infty}[1/a(t)]\int_{t_0}^{t}\left[r_0(s)+r_1(s)+r_2(s)\big(q(s)/a(s)\big)^{m/2}+w_1(s)\big(q(s)/a(s)\big)^{1/2}\right]ds\,dt<\infty,$$

then (1) has a nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty}x(t)>0$.

Another variant of Theorem 8 is as follows.

Theorem 9. *Let* (3)—(8) *and* (11) *be satisfied. If*

$$\int_{t_0}^{\infty}|q'(s)|\,\big(a(s)/q^3(s)\big)^{1/2}\,ds<\infty \tag{27}$$

$$\int_{t_0}^{\infty}[sq(s)/a(s)]\,ds=\infty, \tag{28}$$

$$\int_{t_0}^{\infty}\left[\big(r_0(s)+r_1(s)+r_2(s)(q(s)/a(s))^{m/2}\big)/q(s)\right]ds<\infty, \tag{29}$$

and

$$\int_{t_0}^{t}\left[w_1(s)/\big(a(s)\,q(s)\big)^{1/2}\right]ds=o(t),\quad t\to\infty, \tag{30}$$

then every nonoscillatory or Z-type solution $x(t)$ of (1) *satisfies $x(t)\to 0$ as $t\to\infty$.*

Proof. Let $x(t)$ be a nonoscillatory or Z-type solution of (1). We first show that $\liminf_{t\to\infty}|x(t)|=0$. Suppose that $x(t)\geqq 0$ for $t\geqq t_1\geqq t_0$ and that $\liminf_{t\to\infty}x(t)>0$. Let t_2, L, and δ be as in the proof of Theorem 8 and proceed as in that proof until (26) is obtained. Dividing (26) by $q(t)$, integrating, and then using (27) and (29) we obtain

$$\begin{aligned}a(t)\,y(t)/q(t)&\leqq a(t_2)\,y(t_2)/q(t_2)+L\int_{t_2}^{t}|q'(s)|\big(a(s)/q^3(s)\big)^{1/2}\,ds+\\&+\int_{t_2}^{t}\left[\big(r_0(s)+Lr_1(s)+L^m r_2(s)(q(s)/a(s))^{m/2}\big)/q(s)\right]ds+\\&+L\int_{t_2}^{t}\left[w_1(s)/\big(a(s)\,q(s)\big)^{1/2}\right]ds-\delta(t-t_2)\leqq\\&\leqq M_1-\delta t+L\int_{t_2}^{t}\left[w_1(s)/\big(a(s)\,q(s)\big)^{1/2}\right]ds\end{aligned}$$

for some positive constant M_1. Multiplying the last inequality by $q(t)/a(t)$ and then integrating we have

$$x(t) \leqq x(t_2)+\int_{t_2}^{t}\left\{\left[M_1/s-\delta+(L/s)\int_{t_2}^{s} w_1(u)/\big(a(u)q(u)\big)^{1/2}\,du\right] sq(s)/a(s)\right\} ds.$$

From (30) there exists $t_3 \geqq t_2$ such that

$$M_1/s-\delta+(L/s)\int_{t_2}^{s} w_1(u)/\big(a(u)q(u)\big)^{1/2}\,du < -\delta/2$$

for $s \geqq t_3$. Hence there exists a constant $M_2>0$ such that

$$x(t) < M_2-(\delta/2)\int_{t_3}^{t}[sq(s)/a(s)]\,ds$$

and (28) implies that $x(t)<0$ for all sufficiently large t, which is a contradiction. The remainder of the proof is the same as the proof of Theorem 8.

Remark. Theorem 8 includes Theorem 5 in [4] and it generalizes Theorem 2.2 in [8]. Theorems 8 and 9 extend Theorem 3 in [14].

Finally, consider the equation

$$(x'/2t)'+2tx = 2/t^3+12/t^7 \tag{31}$$

having the general solution

$$x(t) = A\sin t^2+B\cos t^2+1/t^4.$$

We can conclude from either Theorem 8 or Theorem 9 that the nonoscillatory and Z-type solutions of (31) tend to zero as $t\to\infty$. However, none of Theorems 4—5 nor Corollaries 6—7 apply to this example since $\big(a(t)q(t)\big)'\equiv 0$ and $w(t)\leqq 0$ which prevents (13) from being satisfied. Notice that none of the oscillatory solutions of (31) converge to zero as $t\to\infty$. In general it can be easily shown that in the case $h\equiv 0$ under conditions (3)—(8) all the oscillatory solutions of equation (1) tend to zero as $t\to\infty$ only if

$$\int_{t_0}^{\infty}\left[\left|\big(a(s)q(s)\big)'\right|/a(s)q(s)\right] ds = \infty.$$

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[4] Research Supported by the Mississippi State University Biological and Physical Sciences Research Institute

[5] Research Supported by Hungarian National Foundation for Scientific Research Grant No. 6032/6319

(Received Jun. 5, 1987)