

Lacunary spline interpolation in L^p

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Recently in [3] TH. FAWZY and F. HOLAIL have presented a $(0, 2)$ -type lacunary interpolation method by splines of degree 6. Here we show that the same method can be applied for approximation in L^p , if the function values and the second derivatives are replaced by integral mean values and second order difference quotients, respectively. Here we present our results in $L^p(\mathbf{R})$, but we remark, that similar results can be proved in the case of $L^p(a, b)$, where (a, b) is any finite open interval. Our main ideas are similar to those of [4].

For any integer $r \geq 0$ the space W_p^r with $1 \leq p < \infty$ consists of all r -times differentiable functions with an r -th derivative belonging to L^p . The L^p -modulus of continuity $\omega_r(f, h)_p$ for any f in L^p is defined as usually ($h > 0$) (see e.g. [1], [2]).

The following theorem — which is a slight modification of a result in [1] — has been proved in [4].

Lemma. *Let Γ be an arbitrary set and $L_\gamma: L^p \rightarrow L^p$ ($1 \leq p < \infty$) for any γ in Γ , uniformly bounded linear operators, for which there exist a function $a: \Gamma \rightarrow [0, 1]$, an integer $r \geq 1$ and a constant $c > 0$ with the property that*

$$\|L_\gamma(f) - f\|_p \leq ca(\gamma) \|f^{(r)}\|_p$$

whenever f is in W_p^r . Then there exists a constant $d > 0$ with

$$\|L_\gamma(f) - f\|_p \leq d\omega_r(f, a(\gamma)^{1/r})_p$$

whenever f is in L^p .

Let $f \in L^p$ be arbitrary and we define

$$f_h(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t) dt.$$

The function f_h is called the Steklov-transform of f . Let $h > 0$ and $\{x_k\}$ be a subdivision of $\mathbf{R}$ with $x_{k+1} - x_k = h$. Let for all k

$$f_k = f_h(x_k), \quad f_k'' = \frac{1}{h^2} \cdot \Delta^2 f_k$$

and

$$S_h(x) = \sum_{j=0}^6 \frac{S_k^{(j)}}{j!} \cdot (x-x_k)^j, \quad x_k \equiv x \equiv x_{k+1}$$

where

$$S_k^{(0)} = f_k, \quad S_k^{(2)} = f_k'',$$

$$S_k^{(6)} = \frac{1}{h^4} \cdot \Delta^4 f_{k-1}'',$$

$$S_k^{(5)} = \frac{1}{h^3} \cdot \Delta^3 f_{k-1}'' - \frac{h}{2} S_k^{(6)},$$

$$S_k^{(4)} = \frac{1}{h^2} \cdot \Delta^2 f_{k-1}'' - \frac{h^2}{12} S_k^{(6)},$$

$$S_k^{(3)} = \frac{1}{h} \cdot \Delta f_k - \sum_{r=2}^4 \frac{h^{r-1}}{r!} S_k^{(r+2)},$$

$$S_k^{(1)} = \frac{1}{h} \cdot \Delta f_k - \frac{h}{2} f_k'' - \sum_{r=3}^6 \frac{h^{r-1}}{r!} S_k^{(r)}.$$

Then this S_h is a continuous function with a right continuous second derivative, and it is a piecewise polynomial of degree 6 (see [3]).

Theorem. For any f in L^p ($1 \leq p < \infty$) we have

$$\|S_h - f\|_p \leq \text{const } \omega_2(f, h)_p$$

for $0 < h \leq 1$.

PROOF. It is trivial that the operator $f \rightarrow S_h$ is linear. In order to apply the lemma we first prove the uniform boundedness.

In what follows we shall often use the inequality

$$\left| \int_E f \right|^p \leq m(E)^{p-1} \int_E |f|^p, \quad (f \in L^p(E), 1 \leq p < \infty).$$

We have

$$\begin{aligned} \|S_h\|_p &\leq \sum_{j=0}^6 \left[\sum_k \int_{x_k}^{x_{k+1}} \left| \frac{S_k^{(j)}}{j!} (x-x_k)^j \right|^p dx \right]^{1/p} = \\ &= \sum_{j=0}^6 \frac{h^{j+1/p}}{j! (pj+1)^{1/p}} \left[\sum_k |S_k^{(j)}|^p \right]^{1/p}. \end{aligned}$$

As $S_k^{(j)}$ is a linear combination of terms of the type $\frac{1}{h^j} f_k$, we first estimate the quantity

$$\begin{aligned} \left[\left| \frac{1}{h^j} f_k \right|^p \right]^{1/p} &= \frac{1}{h^j} \left[\sum_k \left| \frac{1}{h} \cdot \int_{x_k-h/2}^{x_k+h/2} f(x) dx \right|^p \right]^{1/p} \leq \\ &\leq \frac{1}{h^j} \left[\sum_k \frac{1}{h^p} h^{p-1} \int_{x_k-h/2}^{x_k+h/2} |f(x)|^p dx \right]^{1/p} = h^{-j-1/p} \|f\|_p \end{aligned}$$

and, by the above estimation, we get

$$\|S_h\|_p \leq \text{const } \|f\|_p$$

which proves the uniform boundedness.

Now let $f \in W_p^2$. By the Taylor-formula we have

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \int_{x_k}^x (x - \xi) f''(\xi) d\xi$$

where $x_k < \xi < x$ and hence for all $x_k \leq x \leq x_{k+1}$

$$\begin{aligned} (1) \quad S_h(x) - f(x) &= [f_k - f(x_k)] + \left[\left(\frac{1}{h} \cdot \Delta f_k - f'(x_k) \right) (x - x_k) \right] - \\ &- \left[\frac{h}{2} f_k''(x - x_k) \right] - \left[\sum_{r=3}^6 \frac{h^{r-1}}{r!} S_k^{(r)}(x - x_k) \right] + \left[\frac{1}{2} f_k''(x - x_k)^2 \right] - \left[\int_{x_k}^x (x - \xi) f''(\xi) d\xi \right] + \\ &+ \left[\frac{1}{3!} S_k^{(3)}(x - x_k)^3 \right] + \left[\frac{1}{4!} S_k^{(4)}(x - x_k)^4 \right] + \left[\frac{1}{5!} S_k^{(5)}(x - x_k)^5 \right] + \left[\frac{1}{6!} S_k^{(6)}(x - x_k)^6 \right]. \end{aligned}$$

It is enough to prove that all terms in the square brackets have L^p -norms which are not greater than $\text{const } h^2 \|f''\|_p$.

First of all we estimate the quantity $\left[\sum_k \left| \frac{1}{h^2} \cdot \Delta^2 f_k \right|^p \right]^{1/p}$.

$$\begin{aligned} \sum_k \left| \frac{1}{h^2} \cdot \Delta^2 f_k \right|^p &= \sum_k \left| \frac{1}{h} \left[\frac{f_h(x_{k+2}) - f_h(x_{k+1})}{h} - \frac{f_h(x_{k+1}) - f_h(x_k)}{h} \right] \right|^p = \\ &= \sum_k \left| \frac{f_h'(\xi_{k+1}) - f_h'(\xi_k)}{h} \right|^p = \sum_k \left| \frac{1}{h} \int_{\xi_k}^{\xi_{k+1}} f_h''(\xi) d\xi \right|^p \leq \\ &\leq \frac{1}{h^p} \sum_k (\xi_{k+1} - \xi_k)^{p-1} \int_{x_k}^{x_{k+2}} |f_h''(\xi)|^p d\xi \leq \frac{1}{h^p} (2h)^{p-1} 2 \|f_h''\|_p^p = \frac{2^p}{h} \|f_h''\|_p^p, \end{aligned}$$

where $x_k < \xi_k < x_{k+1} < \xi_{k+1} < x_{k+2}$. But

$$\begin{aligned} \|f_h''\|_p &= \left[\int |f_h''(x)|^p dx \right]^{1/p} = \left[\int \left| \frac{1}{h} f' \left(x + \frac{h}{2} \right) - \frac{1}{h} f' \left(x - \frac{h}{2} \right) \right|^p dx \right]^{1/p} = \\ &= \frac{1}{h} \left[\int \left| f' \left(x + \frac{h}{2} \right) - f' \left(x - \frac{h}{2} \right) \right|^p dx \right]^{1/p} = \frac{1}{h} \left[\int \left| \int_{x-h/2}^{x+h/2} f''(t) dt \right|^p dx \right]^{1/p} \leq \\ &\leq \frac{1}{h} \left[\int h^{p-1} \int_{x-h/2}^{x+h/2} |f''(t)|^p dt dx \right]^{1/p} = \frac{1}{h} \left[h^{p-1} \sum_k \int_{x_k}^{x_{k+1}} \int_{x-h/2}^{x+h/2} |f''(t)|^p dt dx \right]^{1/p} \leq \\ &\leq \frac{1}{h} \left[h^{p-1} \sum_k \int_{x_k}^{x_{k+1}} \int_{x_k-h/2}^{x_{k+1}+h/2} |f''(t)|^p dt dx \right]^{1/p} = \frac{1}{h} \left[h^p \sum_k \int_{x_k-h/2}^{x_{k+1}+h/2} |f''(t)|^p dt \right]^{1/p} = \\ &= 2 \|f''\|_p \end{aligned}$$

and hence

$$\left[\sum_k \left| \frac{1}{h^2} \cdot \Delta^2 f_k \right|^p \right]^{1/p} \cong \frac{4}{h^{1/p}} \|f''\|_p.$$

On the other hand, it is easy to see, that $S_k^{(r)}$ ($r \geq 3$) is a linear combination of terms of the type

$$\frac{1}{h^{r-2}} \left[\frac{1}{h^2} \cdot \Delta^2 f_k \right],$$

which implies that for $r \geq 3$

$$\left[\sum_k |S_k^{(r)}|^p \right]^{1/p} \cong \text{const} \frac{1}{h^{r-2+1/p}} \|f''\|_p.$$

From this fact we infer that for $j \geq 3$

$$\left[\sum_k \int_{x_k}^{x_{k+1}} \left| \frac{1}{j!} S_k^{(j)} (x - x_k)^j \right|^p dx \right]^{1/p} = \frac{1}{j!} \left[\frac{h^{pj+1}}{pj+1} \sum_k |S_k^{(j)}|^p \right]^{1/p} \cong \text{const } h^2 \|f''\|_p,$$

which is the desired estimation for the L^p -norm of the last four terms in (1).

For the first term we have

$$\begin{aligned} \left[\sum_k \int_{x_k}^{x_{k+1}} |f_k - f(x_k)|^p dx \right]^{1/p} &= \left[h \sum_k \left| \frac{1}{h} \int_{x_k-h/2}^{x_k+h/2} [f(x) - f(x_k)] dx \right|^p \right]^{1/p} = \\ &= \left[h \sum_k \left| \frac{1}{h} \int_{x_k-h/2}^{x_k+h/2} \left[f'(x_k)(x - x_k) + \int_{x_k}^x (x - \xi) f''(\xi) d\xi \right] dx \right|^p \right]^{1/p} = \\ &= \left[h \sum_k \left| \frac{1}{h} \int_{x_k-h/2}^{x_k+h/2} \int_{x_k}^x (x - \xi) f''(\xi) d\xi dx \right|^p \right]^{1/p} = \\ &\cong \left[h^{1-p} \sum_k h^{p-1} \int_{x_k-h/2}^{x_k+h/2} \left[\int_{x_k}^x |(x - \xi) f''(\xi)|^p d\xi \right] dx \right]^{1/p} \cong \\ &\cong \left[\sum_k h^p \int_{x_k-h/2}^{x_k+h/2} \left[\int_{x_k}^{x_k+h/2} |f''(\xi)|^p d\xi \right] dx \right]^{1/p} \cong \\ &\cong \left[\sum_k h^{p+1} h^{p-1} \int_{x_k-h/2}^{x_k+h/2} |f''(\xi)|^p d\xi \right]^{1/p} = h^2 \|f''\|_p. \end{aligned}$$

For the second term we obtain

$$\begin{aligned} \left[\sum_k \int_{x_k}^{x_{k+1}} \left| \frac{1}{h} \cdot \Delta f_k - f'(x_k) \right|^p (x - x_k)^p dx \right]^{1/p} &= \\ &= \left[\frac{h^{p+1}}{p+1} \sum_k \left| \frac{1}{h^2} \int_{x_k-h/2}^{x_k+h/2} [f(x+h) - f(x) - f'(x_k)h] dx \right|^p \right]^{1/p}. \end{aligned}$$

On the other hand, by the Taylor-formula we get

$$f(x+h) = f(x_k) + f'(x_k)(x+h-x_k) + \int_{x_k}^{x+h} (x+h-\xi) f''(\xi) d\xi,$$

$$f(x) = f(x_k) + f'(x_k)(x-x_k) + \int_{x_k}^x (x-\xi) f''(\xi) d\xi$$

and hence

$$f(x+h) - f(x) - f'(x_k)h = \int_x^{x+h} (x-\xi) f''(\xi) d\xi + h \int_{x_k}^{x+h} f''(\xi) d\xi$$

that is, continuing the above estimation

$$\begin{aligned} & \left[\frac{h^{p+1}}{p+1} \sum_k \left| \frac{1}{h^2} \int_{x_k-h/2}^{x_k+h/2} \int_x^{x+h} (x-\xi) f''(\xi) d\xi dx + \frac{1}{h} \int_{x_k-h/2}^{x_k+h/2} \int_{x_k}^{x+h} f''(\xi) d\xi dx \right|^p \right]^{1/p} \equiv \\ & \equiv \left[\frac{2^p h^{p+1}}{p+1} \sum_k \left\{ \frac{h^{p-1}}{h^{2p}} \int_{x_k-h/2}^{x_k+h/2} \left| \int_x^{x+h} (x-\xi) f''(\xi) d\xi \right|^p dx + \right. \right. \\ & \quad \left. \left. + \frac{h^{p-1}}{h^p} \int_{x_k-h/2}^{x_k+h/2} \left| \int_{x_k}^{x+h} f''(\xi) d\xi \right|^p dx \right\} \right]^{1/p} \equiv \\ & \equiv \left[\frac{2^p h^{p+1}}{p+1} \sum_k \left\{ \frac{h^p}{h^{p+1}} \int_{x_k-h/2}^{x_k+h/2} h^{p-1} \int_{x_k-h/2}^{x_k+3h/2} |f''(\xi)|^p d\xi dx + \right. \right. \\ & \quad \left. \left. + \frac{1}{h} \int_{x_k-h/2}^{x_k+h/2} \left(\frac{3h}{2} \right)^{p-1} \int_{x_k-h/2}^{x_k+3h/2} |f''(\xi)|^p d\xi dx \right\} \right]^{1/p} \equiv \text{const } h^2 \|f''\|_p, \end{aligned}$$

where we applied the inequality $|a+b|^p \leq 2^p(|a|^p + |b|^p)$. The estimation for the sixth term in (1) is as follows:

$$\begin{aligned} & \left[\sum_k \int_{x_k}^{x_{k+1}} \left| \int_{x_k}^x (x-\xi) f''(\xi) d\xi \right|^p dx \right]^{1/p} \equiv \left[\sum_k \int_{x_k}^{x_{k+1}} \left[\int_{x_k}^x (x-\xi) |f''(\xi)| d\xi \right]^p dx \right]^{1/p} \equiv \\ & \equiv h \left[\sum_k \int_{x_k}^{x_{k+1}} \left[\int_{x_k}^{x_{k+1}} |f''(\xi)| d\xi \right]^p dx \right]^{1/p} \equiv h \left[h \sum_k h^{p-1} \int_{x_k}^{x_{k+1}} |f''(\xi)|^p d\xi \right]^{1/p} = h^2 \|f''\|_p, \end{aligned}$$

and for the fifth term of (1):

$$\begin{aligned} & \left[\sum_k \int_{x_k}^{x_{k+1}} |f''_k(x-x_k)|^p dx \right]^{1/p} = \left[\frac{h^{2p+1}}{2p+1} \sum_k \left| \frac{1}{h^2} \cdot \Delta^2 f_k \right|^p \right]^{1/p} \equiv \\ & \equiv \frac{h^{2+1/p}}{(2p+1)^{1/p}} \frac{4}{h^{1/p}} \|f''\|_p = \text{const } h^2 \|f''\|_p. \end{aligned}$$

Similarly

$$\left[\sum_k \int_{x_k}^{x_{k+1}} |h f''_k(x-x_k)|^p dx \right]^{1/p} \equiv \text{const } h^2 \|f''\|_p.$$

Finally, the estimation for the fourth term in (1) is very similar to that of the last terms, and we have

$$\|S_h - f\|_h \leq \text{const } h^2 \|f''\|_p.$$

Now applying the lemma we get the statement.

References

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