

Characterization of pairs of additive functions with some regularity property

By Z. DARÓCZY and I. KÁTAI

Dedicated to Prof. Béla Gyires on his 80th birthday

1. For a group G let $\mathcal{A}_G^*$ denote the set of completely additive functions $f: \mathbb{N} \rightarrow G$, i.e. those functions for which $f(mn) = f(m) + f(n)$, $\forall m, n \in \mathbb{N}$.

Let $\mathcal{K}$ be the set of all infinite monotonically increasing sequences of positive integers. For typical elements $N = \{n_1 < n_2 < \dots\}$, $M = \{m_1 < m_2 < \dots\}$ of $\mathcal{K}$ let $N \oplus M = \{n_1 + m_1 < n_2 + m_2 < \dots\}$, $N * M = \{n_1 m_1 < n_2 m_2 < \dots\}$. Let furthermore for $k \in \mathbb{Z}$, $k \oplus N$ denote the sequence of the positive elements of $k + n_j$ ($j = 1, 2, \dots$), and let $kN = \{kn_1 < kn_2 < \dots\}$ for $k \in \mathbb{N}$.

In what follows under a topological group we always mean a T_0 -group, that guarantees that the metric can be chosen to be translation invariant (see [4]).

Let G_1, G_2 be metrically compact Abelian groups, $\varphi \in \mathcal{A}_{G_1}^*$, $\psi \in \mathcal{A}_{G_2}^*$. Let $\mathcal{K}_\varphi$ (resp. $\mathcal{K}_\psi$) be the set of those $N \in \mathcal{K}$ for which the limit $a_\varphi(N) := \lim_j \varphi(n_j)$ (resp. $a_\psi(N) := \lim_j \psi(n_j)$) exists.

We should like to determine all those pairs of functions (φ, ψ) for which $N \in \mathcal{K}_\varphi$ implies that $-1 \oplus N \in \mathcal{K}_\psi$. The case $G_1 = G_2$, $\varphi = \psi$ under the similar condition $N \in \mathcal{K}_\varphi \Rightarrow 1 \oplus N \in \mathcal{K}_\varphi$ has been considered in our earlier papers [1], [2]. It was proved

Lemma 1. *If G is a metrically compact Abelian group, $\varphi \in \mathcal{A}_G^*$, with the property $N \in \mathcal{K}_\varphi \Rightarrow 1 \oplus N \in \mathcal{K}_\varphi$, then there exists a continuous homomorphism $F: \mathbb{R}_x \rightarrow G$, $\mathbb{R}_x$ denotes the multiplicative group of the positive reals, such that φ is a restriction of F on the set $\mathbb{N}$. Conversely, if $F: \mathbb{R}_x \rightarrow G$ is a continuous homomorphism, then $\varphi(n) := F(n)$ ($\forall n \in \mathbb{N}$) is such a function, for which $N \in \mathcal{K}_\varphi$ implies $1 \oplus N \in \mathcal{K}_\varphi$.*

In the proof we used the next theorem due to WIRSING [3] which will be formulated now as

Lemma 2. *Let T denote the one-dimensional torus, $\varphi \in \mathcal{A}_T^*$, $\Delta\varphi(n) := \varphi(n+1) - \varphi(n) \rightarrow 0$ ($n \rightarrow \infty$). Then there exists a $\tau \in \mathbb{R}$ such that*

$$\varphi(n) \equiv \tau \log n \pmod{1}.$$

2. The following assertions can be proved easily, arguing as it was done in [1], [2].

Lemma 3. *Let K_φ , resp. K_ψ denote the set of all accumulation points of $\{\varphi(n) | n \in \mathbb{N}\}$, resp. $\{\psi(n) | n \in \mathbb{N}\}$. Then $K_\varphi \subseteq G_1$, $K_\psi \subseteq G_2$, and $\varphi(n) \in K_\varphi$, $\psi(n) \in K_\psi$ ($\forall n \in \mathbb{N}$).*

Lemma 4. Let $\varphi \in \mathcal{A}_{G_1}^*$, $\psi \in \mathcal{A}_{G_2}^*$ and assume that $N \in \mathcal{K}_\varphi$ implies that $k \oplus N \in \mathcal{K}_\psi$, with a given $k \in \mathbb{Z}$. Then there exists a function $H: K_\varphi \rightarrow K_\psi$ with the following property:

If $N \in \mathcal{K}_\varphi$ and $a_\varphi(N) = g$, then $a_\psi(k \oplus N) = H(g)$. Furthermore $H[K_\varphi] = K_\psi$, H is a continuous function.

Lemma 5. Assume that the conditions of Lemma 4 hold with $k = -1$. Then they hold with $k = 1$, as well.

PROOF. From Lemma 4 we have that for $N \in \mathcal{K}_\varphi$, $a_\varphi(N) = g$, $a_\psi(-1 \oplus N) = L(g)$, where $L: K_\varphi \rightarrow K_\psi$ is a continuous function. By using the additivity of φ and ψ , and the identity $n^2 - 1 = (n-1)(n+1)$, we deduce that $\varphi(n_v) \rightarrow g$, $\varphi(n_v^2) \rightarrow 2g$, $\psi(n_v - 1) \rightarrow L(g)$, $\psi(n_v^2 - 1) \rightarrow L(2g)$, whence $\psi(n_v + 1) = \psi(n_v^2 - 1) - \psi(n_v - 1)$, $\psi(n_v + 1) \rightarrow L(2g) - L(g) =: S(g)$. $\square$

Now and in the sequel we assume that $\varphi(n_v) \rightarrow g \Rightarrow \varphi(n_v - 1) \rightarrow L(g)$. Then, by Lemma 5, $\varphi(n_v + 1) \rightarrow S(g)$, $S(g) = L(2g) - L(g)$. From Lemma 4 we have that L, S are continuous, $L[K_\varphi] = S[K_\varphi] = K_\psi$.

Let the translation invariant metrics defined in G_1, G_2 be ϱ, σ , respectively. For $A, B \subseteq G_1$, $C, D \subseteq G_2$, let

$$\varrho(A, B) = \inf_{x \in A, y \in B} \varrho(x, y); \quad \varrho(C, D) = \inf_{u \in C, v \in D} \sigma(u, v).$$

Let furthermore

$$F_h := \{g \mid g \in K_\varphi, L(g) = h\},$$

$$E_h := \{g \mid g \in K_\varphi, S(g) = h\}.$$

Lemma 6. The sets F_h, E_h are closed for each $h \in K_\psi$.

PROOF. It is clear, since L, S are continuous.

Lemma 7. The condition $\psi(n_v) \rightarrow h$ holds for some $h \in K_\psi$ if and only if $\varrho(\varphi(n_v - 1), E_h) \rightarrow 0$ ($v \rightarrow \infty$).

PROOF. Assume that $\psi(n_v) \rightarrow h$ and $\varrho(\varphi(n_v - 1), E_h) \rightarrow 0$. Then

$$\varrho(\varphi(n_{v'} - 1), E_h) > \delta (> 0)$$

for a suitable infinite subsequence $n_{v'}$ of n_v . Then for a suitable rarefied subsequence $n_{v''}$ of $n_{v'}$, $\varphi(n_{v''} - 1) \rightarrow \tau$ ($\tau \in K_\varphi$). Then $\varrho(\tau, E_h) > \delta/2$, consequently $\tau \notin E_h$, $\psi(n_{v''}) \rightarrow S(\tau) \neq h$, which is a contradiction. Assume now that $\varrho(\varphi(n_v - 1), E_h) \rightarrow 0$ and $\psi(n_v) \not\rightarrow h$. Then there exists such a rarefied subsequence $n_{v'}$ for which $\psi(n_{v'}) \rightarrow \tau$ ($\tau \neq h$). After a further rarefaction we have $\varphi(n_{v''} - 1) \rightarrow \eta$, $\psi(n_{v''}) \rightarrow \tau$. From the continuity of $\varrho(\cdot, E_h)$ we have $\varrho(\eta, E_h) = 0$, which implies that $\eta \in E_h$, since E_h is closed. $\square$

Lemma 8. The condition $\psi(n_v) \rightarrow h$ holds for some $h \in K_\psi$ if and only if $\varrho(\varphi(n_v + 1), E_h) \rightarrow 0$ ($v \rightarrow \infty$).

PROOF. The same as the proof of Lemma 7.

Lemma 9. If $N \in K_\varphi$, $a_\varphi(N) = 0$, then $a_\psi(N) = 0$.

PROOF. Assume that $\varphi(n_v) \rightarrow 0$. Then $\varphi(n_v^3) = 3\varphi(n_v) \rightarrow 0$, $\psi(n_v + 1) \rightarrow S(0) = L(2 \cdot 0) - L(0) = 0$, $\psi(n_v^3 + 1) \rightarrow 0$. Consequently

$$\psi\left(\frac{n_v^3 + 1}{n_v + 1}\right) = \psi(n_v^2 - n_v + 1) = \psi(n_v(n_v - 1) + 1) \rightarrow 0.$$

From Lemma 7 we have

$$\varrho(\varphi(n_v(n_v - 1)), E_0) \rightarrow 0,$$

and by $\varphi(n_v) \rightarrow 0$, $\varrho(\varphi(n_v - 1), E_0) \rightarrow 0$. Applying Lemma 7 again, we get $\varphi(n_v) \rightarrow 0$. $\square$

Lemma 10. *There exists a continuous homomorphism $H: K_\varphi \rightarrow K_\psi$, such that $N \in \mathcal{K}_\varphi$ implies that $a_\psi(N) = H(a_\varphi(N))$.*

PROOF. Let $g \in K_\varphi$. From Lemma 3 it follows that there exists an $M \in K_\varphi$, such that $a_\varphi(M) = -g$.

After a suitable rarefaction, if it is needed, we can assume that $a_\psi(M)$ exists as well. Let us assume that M is so chosen. Let now N be an arbitrary sequence $N \in \mathcal{K}_\varphi$, such that $a_\varphi(N) = g$. Then $\varphi(n_v m_v) = \varphi(n_v) + \varphi(m_v) \rightarrow g - g = 0$, and so by Lemma 9, $\psi(n_v m_v) \rightarrow 0$. Since $a_\psi(M)$ exists, therefore $N \in \mathcal{K}_\psi$ and $a_\psi(N) = -a_\psi(M)$. This means that $N \in \mathcal{K}_\varphi \Rightarrow N \in \mathcal{K}_\psi$ and that the value $a_\psi(N)$ depends only on g .

If $a_\varphi(N) = g_1$, $a_\varphi(M) = g_2$, then $\varphi(n_v m_v) \rightarrow g_1 + g_2$, and by the additivity of φ and ψ we deduce immediately that $H(g_1 + g_2) = H(g_1) + H(g_2)$. Hence (or from Lemma 9) we have $H(0) = 0$.

Finally, the continuity of H is straightforward, so we omit the proof. $\square$

Lemma 11. *Let $U = \{g | g \in K_\varphi, H(g) = 0\}$. Then U is a compact subgroup of K_φ .*

PROOF. Clear.

3. Let $M \in \mathcal{K}_\varphi$ be a fixed sequence such that $a_\varphi(M) = 0$. Let

$$(3.1) \quad r_1(m, k) := \sup_{n \geq k} \sigma(\psi(nm - 1), \psi(n - 1)),$$

$$(3.2) \quad r_2(m, k) := \sup_{n \geq k} \sigma(\psi(nm + 1), \psi(n + 1)).$$

Lemma 12. *For an arbitrary sequence $k_v \nearrow \infty$, we have, $r_1(m_v, k_v) \rightarrow 0$, $r_2(m_v, k_v) \rightarrow 0$ ($v \rightarrow \infty$).*

PROOF. Assume that $r_1(m_v, k_v) \not\rightarrow 0$. Let $\{v'\}$ be a suitable rarefaction of $\{v\}$, such that $\psi(n_{v'} m_{v'} - 1) - \psi(n_{v'} - 1) \rightarrow \tau \neq 0$. After a further rarefaction $\{v''\}$ of $\{v'\}$ we can assume that $\lim \varphi(n_{v''}) = \eta$ exists. But then $\varphi(n_{v''} m_{v''}) \rightarrow \eta$, consequently $\psi(n_{v''} - 1) \rightarrow L(\eta)$, $\psi(n_{v''} m_{v''} - 1) \rightarrow L(\eta)$, that contradicts to $\tau \neq 0$. So the first assertion is true. The proof of the second assertion is the same, therefore we omit it.

Lemma 13. *We have, $\psi(n + 1) - \psi(n) \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF. In $\sigma(\psi(nm - 1), \psi(n - 1))$ write first $n = k + 1$, then we get $\sigma(\psi(km + n - 1), \psi(km))$, substituting now $k = (m - 1)s$, and applying the translation invariant property of σ , this is the same as $\sigma(\psi(ms + 1), \psi(s))$. So we have $r_3(m_v, k_v) \rightarrow$

$\rightarrow 0$, where

$$(3.3) \quad r_3(m, k) := \sup_{n \geq k} \sigma(\psi(mn+1), \psi(n)).$$

From (3.2), (3.3) it follows immediately, $\psi(n+1) - \psi(n) \rightarrow 0$. $\square$

The condition $\psi(n+1) - \psi(n) \rightarrow 0$ is equivalent with $H(\varphi(n+1) - \varphi(n)) \rightarrow 0$ ($n \rightarrow \infty$).

Let us consider now the factor-group $K_\lambda := K_\varphi/U$, with the natural metric r defined as follows. If $U_g = g_i + U (\in K_\lambda)$, then $r(U_{g_1}, U_{g_2}) = \varrho(g_1 - g_2, U)$. It is clear that K_λ is compact. Let $\lambda \in \mathcal{A}_{K_\lambda}^*$ defined by $\lambda(n) := \varphi(n) + U$. This induces a mapping $B: K_\lambda \rightarrow K_\psi$, which is a topological isomorphism. Indeed, let $B(\varphi(n) + U) = \psi(n) + U = H(\varphi(n)) + U$, and in general $B(U_g) = H(g)$. Since $U_{g_1+g_2} = U_{g_1} + U_{g_2}$, and $H(g_1+g_2) = H(g_1) + H(g_2)$, therefore

$$B(U_{g_1} + U_{g_2}) = B(U_{g_1}) + B(U_{g_2}).$$

It is clear that $U_{g_v} \rightarrow U_h$ if and only if $H(g_v) \rightarrow H(h)$, which guarantees that B and the inverse mapping B^{-1} are continuous. From now on we may assume that $\psi = \lambda$.

Let us assume now that for $\varphi = \varphi_1$, $\varphi_2 \in \mathcal{A}_{G_1}^*$ the conditions $N \in \mathcal{K}_{\varphi_i} \Rightarrow -1 \oplus \oplus N \in \mathcal{K}_\lambda$ hold for $i=1, 2$, and that $K_{\varphi_1} = K_{\varphi_2}$. Then the function $H: K_{\varphi_i} \rightarrow K_\lambda$ is well defined, $H(\varphi_i(n)) = \lambda(n)$, consequently $H(\varphi_1(n) - \varphi_2(n)) = 0 \quad \forall n \in \mathbb{N}$, i.e. $\varphi_1(n) - \varphi_2(n) \in U$, $\varphi_1(n) = \varphi_2(n) + u(n)$, $u \in \mathcal{A}_U^*$.

Let φ, λ be such a pair for which $N \in \mathcal{K}_\varphi \Rightarrow -1 \oplus N \in \mathcal{K}_\lambda$. Then the same condition holds for each pair (φ_1, λ) as well, where $\varphi_1(n) = \varphi(n) + u(n)$, $u \in \mathcal{A}_U^*$, U is defined in Lemma 11.

Since $K_\varphi = K$ is defined as the set of all limit points of sequences $\{\varphi(n_v) | n_1 < n_2 < \dots\}$, therefore its cardinality is not greater than the continuum. Since G_1 is a T_0 group, therefore G_1 and so K is connected.

Since $\lambda \in \mathcal{A}_{K/U}^*$ satisfies the condition $\lambda(n+1) - \lambda(n) \rightarrow 0$, therefore there exists a continuous homomorphism $A: \mathbf{R}_x \rightarrow K/U$ such that $A(n) := \lambda(n) \quad n \in \mathbb{N}$.

Conversely, let $U \subseteq K \subseteq G_1$ be metrically compact Abelian group, G_1 is a T_0 -group. Let $A: \mathbf{R}_x \rightarrow K/U$ be a continuous homomorphism. Let $\lambda(n) := A(n)$, $n \in \mathbb{N}$. Then $\lambda \in \mathcal{A}_{K/U}^*$, and the condition $A\lambda(n) = \lambda(n+1) - \lambda(n) \rightarrow 0$ holds. Let $\mathcal{P} = \{p\}$ be the set of primes. We define the value $\varphi_1(p)$ by choosing an arbitrary element $h \in U_g$ where U_g is the coset that corresponds to $\lambda(p)$. Then we shall define $\varphi_1(n)$ for composite integers to be a completely additive function in K . Let $H: K \rightarrow K/U$ be the natural homomorphism, $H(g) = U_g$. Then we have $H(\varphi_1(n)) = \lambda(n)$, and the condition $N \in \mathcal{K}_{\varphi_1} \Rightarrow -1 \oplus N \in \mathcal{K}_\lambda$ obviously holds.

Collecting our results we get the following

Theorem. Let G_1, G_2 be metrically compact Abelian groups, G_1 be a T_0 group.

Let $\varphi \in \mathcal{A}_{G_1}^*$, $\psi \in \mathcal{A}_{G_2}^*$ be such functions for which $N \in \mathcal{K}_\varphi$ implies that $-1 \oplus N \in \mathcal{K}_\psi$. Then $\psi(n+1) - \psi(n) \rightarrow 0$ ($n \rightarrow \infty$). Let U be the same as in Lemma 11. Let $\lambda(n) := \varphi(n) + U$. Then $\lambda \in \mathcal{A}_{K_\varphi/U}^*$, furthermore K_φ/U and K_ψ are topologically isomorphic, $\lambda(n)$ corresponds to $\psi(n) \quad \forall n \in \mathbb{N}$. There exists a continuous homomorphism $A: \mathbf{R}_x \rightarrow K_\varphi/U$ such that $\lambda(n) = A(n) \quad (\forall n \in \mathbb{N})$. Furthermore for the natural homomorphism $H: K_\varphi \rightarrow K_\varphi/U$ we have $H(\varphi(n+1) - \varphi(n)) = \lambda(n+1) - \lambda(n) \rightarrow 0$.

Conversely, let $U \subseteq K(\subseteq G_1)$ be compact subgroups of G_1 . Let $\Lambda: \mathbf{R}_x \rightarrow K/U$ be a continuous homomorphism; $\lambda(n) := \Lambda(n)$. Let $\varphi(p)$ be any element of the coset $g+U=\lambda(p)$, for each prime p . Extend the domain of φ to the set n such that $\varphi \in \mathcal{A}_K^*$. Then $H(\varphi(n+1)-\varphi(n))=\lambda(n+1)-\lambda(n) \rightarrow 0$.

Consequently, $N \in \mathcal{K}_\psi \Rightarrow -1 \oplus N \in \mathcal{K}_\lambda$.

The special case $G_1=T$ is formulated as

Corollary. Let T denote the one-dimensional torus, and G_2 be an arbitrary metrically compact Abelian group. Let $\varphi \in \mathcal{A}_T^*$, $\psi \in \mathcal{A}_{G_2}^*$.

Assume that $N \in \mathcal{K}_\varphi \Rightarrow -1 + N \in \mathcal{K}_\psi$. Assume furthermore that $\psi(n)$ is not identically zero.

Then there exist $\tau \in \mathbf{R}$, $M \in \mathbf{N}$, $u \in \mathcal{A}_{\mathbf{Z}_M}^*$ such that $\varphi(n) = \frac{\tau}{M} \log n + u(n) \pmod{1}$.

Let $\lambda(n) = M\varphi(n)$.

Then the correspondence $\lambda(n) \leftrightarrow \psi(n)$ ($\forall n \in \mathbf{N}$) generate a topological isomorphism between K_λ and K_ψ . The converse assertion is true as well.

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Z. DARÓCZY
DEPARTMENT OF MATHEMATICS
L. KOSSUTH UNIVERSITY DEBRECEN
DEBRECEN 10, HUNGARY

I. KÁTAI
EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BUDAPEST, BOGDÁNFY U. 10/b.
H-1117

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