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Partially ordered sets with self–complementary comparability graphs

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Abstract. A 2–poset $(X, \{P, Q\})$ is a pair consisting of a set X and a set $\{P, Q\}$ of two partial order relations on X such that any two distinct elements of X are comparable in exactly one of these relations. We consider 2–posets $(X, \{P, Q\})$ with the property that there exists an order–isomorphism $f : (X, P) \to (X, Q)$. Thus the poset (X, P) has the property that its comparability graph is self–complementary. We derive results about the structure of such 2–posets, and we determine properties of the order–isomorphism f.

1. Introduction.

A *poset* (X, P) is a pair consisting of a set X and a partial order relation P on X. In [1], the author introduced the concept of a multiposet. A *multiposet* (X, p) is a pair consisting of a set X and a set p of partial order relations on X such that for every $x, y \in X$ with $x \neq y$ there exists one, and only one, relation $P \in p$ such that xPy or yPx. A multiposet (X, p) with $|p| = n$ is also called an *n–poset.* An isomorphism f of two posets (X, P) and (Y, Q) is a bijection $f : X \to Y$ such that for $x, x' \in X$ we have xPx' if and only if $(xf)Q(x'f)$.

Self–complementary graphs were introduced by GERHARD RINGEL [3] and HORST SACHS [4]. In this paper, we shall consider self–complementary comparability graphs. Note that the comparability graph of a poset (X, P) is self–complementary if and only if there exists a non–trivial partial order Q on X such that $(X, \{P, Q\})$ is a 2–poset and an isomorphism $f : (X, P) \to (X, Q)$. Note that every such poset has to have dimension two, as a poset (X, P) has dimension 2 if and only if there exists a non–trivial partial order Q such that $(X, \{P, Q\})$ is a 2–poset (see [2], Theorem 3.61).

2. Simple properties

We first ask if there are some conditions for the existence of 2–posets with isomorphic partial orders.

Proposition 2.1. *Let $n > 1$ be a natural number. Then there exists a 2–poset $(X, \{P, Q\})$ with an isomorphism $f : (X, P) \to (X, Q)$ and $|X| = n$ if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.*

PROOF. If there exists such a 2–poset then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ by (A) in [4]. Conversely, let $n \equiv 0 \pmod 4$. Let $m = n/4$, and $X = \{1, 2, \dots, m\} \times \{1, 2, 3, 4\}$. We define P and Q as follows. Let $(x, y), (x', y') \in X$. We have $(x, y)P(x', y')$ if and only if one of the following holds.

(i) $x = x'$ and $y = y'$,
(ii) $(y, y') \in \{(1, 2), (1, 3), (4, 3)\}$,
(iii) $y = y' = 1$ and $x \leq x'$,
(iv) $y = y' = 3$ and $x \geq x'$.

Similarly, we have $(x, y)Q(x', y')$ if and only if one of the following holds.

(i) $x = x'$ and $y = y'$,
(ii) $(y, y') \in \{(1, 4), (2, 3), (2, 4)\}$,
(iii) $y = y' = 2$ and $x \leq x'$,
(iv) $y = y' = 4$ and $x \geq x'$.

We define f by $(x, y)f = (x, y + 1)$ for $y < 4$ and $(x, 4)f = (x, 1)$. Then it is easy to see that $(X, \{P, Q\})$ is a 2–poset and f is an isomorphism $(X, P) \to (X, Q)$.

If $n \equiv 1 \ (mod\, 4)$ we take X as above with an additional element ∞. We define P, Q, f as above with the additional relations $\infty P \infty$, $\infty Q \infty$, $(x, 1)P\infty$, $\infty P(x, 3)$, $(x, 2)Q\infty$, $\infty Q(x, 4)$ for all $x \in \{1, 2, \dots, m\}$, and $\infty f = \infty$. Again, with this definition the proposition follows. □

Proposition 2.2. *Let $(X, \{P, Q\})$ be a 2–poset with an isomorphism $f : (X, P) \to (X, Q)$.*

(a) (X, P) is connected (and hence also (X, Q)).
(b) There exists at most one $x \in X$ with $xf = x$.

Both result hold for self–complementary graphs in general [3], [4]. The proof of the following lemma is easy and shall be left to the reader.

Lemma 2.3. *Let $(X, \{P, Q\})$ be a 2–poset with an isomorphism $f : (X, P) \to (X, Q)$. We define $A = \{x \in X \mid xP(xf)\}$, $B = \{x \in X \mid xQ(xf)\}$, $C = \{x \in X \mid (xf)Px\}$, and $D = \{x \in X \mid (xf)Qx\}$. Then $Af = B$, $Cf = D$, $Bf \subseteq A \cup C$, $Df \subseteq A \cup C$. Furthermore either $\{A, B, C, D\}$ is a partition of X or there exists a fixed point z of f such that $\{\{z\}, A \setminus \{z\}, B \setminus \{z\}, C \setminus \{z\}, D \setminus \{z\}\}$ is a partition of X.*

3. Isomorphisms whose square is an automorphism

If $(X, \{P, Q\})$ is a 2–poset with an isomorphism $f : (X, P) \to (X, Q)$ then it is clear that f^2 is an automorphism of the comparability graph of (X, P), but in general it is not an automorphism of (X, P). We shall now consider the case when it is.

Lemma 3.1. *Let $(X, \{P, Q\})$ be a 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Then the following are equivalent.*
(1) f is an isomorphism $(X, Q) \to (X, P)$.
(2) f^2 is an automorphism of (X, P).
(3) f^2 is an automorphism of (X, Q).

We leave the proof of this to the reader. We now can describe the orbits of f.

Theorem 3.2. *Let $(X, \{P, Q\})$ be a 2–poset and f an isomorphism $(X, P) \to (X, Q)$ and $(X, Q) \to (X, P)$. If Z is an orbit of f on X with $|Z| > 1$ then Z is infinite and there exists $x \in Z$ such that exactly one of the following holds for $n, m \in \mathbb{Z}$ with $n < m$.*
(1) If $n \equiv 0 \pmod 2$ then $(xf^n)P(xf^m)$, otherwise $(xf^n)Q(xf^m)$.
(2) If $m \equiv 1 \pmod 2$ then $(xf^n)P(xf^m)$, otherwise $(xf^n)Q(xf^m)$.
(3) If $n \equiv 0 \pmod 2$ then $(xf^m)P(xf^n)$, otherwise $(xf^m)Q(xf^n)$.
(4) If $m \equiv 1 \pmod 2$ then $(xf^m)P(xf^n)$, otherwise $(xf^m)Q(xf^n)$.

PROOF. There exists an element $x \in Z$ such that x and xf are P–comparable. First suppose that $xP(xf)$. Then we have $(xf)Q(xf^2)$, and hence we can have neither $(xf^2)Px$ nor $(xf^2)Qx$. Assume that $xP(xf^2)$. Using the fact that f^2 is an isomorphism of (X, P), we get $xP(xf^n)$ for all $n \geq 1$, and by the same arguments it follows that we have (1). Similarly, if $xQ(xf^2)$ then we get (2). If $(xf)Px$ then using the same arguments we get (3) if $(xf^2)Px$ and (4) if $(xf^2)Qx$. We finally show that Z is infinite. Note that as $|Z| > 1$, we have $xf \neq x$. But as x and xf are P–comparable, we have xf and xf^2 being Q–comparable, hence $xf^2 \neq x$. Thus x and xf^2 are either P–comparable or Q–comparable, and as $f^2 \in Aut(X, P)$ and $f^2 \in Aut(X, Q)$, we get that $xf^{2n} \neq xf^{2m}$ whenever, $n, m \in \mathbb{Z}$ with $n \neq m$, and hence Z is infinite. □

Thus we have seen that there are only 4 different possibilities for the orbits of f on X. We remark that with methods similar to those used in Theorem 3.2 one can also find restrictions for the relations between elements of two distinct orbits. Finally note that if A, B, C, D are defined as in Lemma 2.3 then we have $Bf = A$ and $Df = C$.

4. Finite 2–posets

The general theory of self–complementary graphs gives some information about the action of f on X, for example, [4], (B) shows that the lenght of its orbits is 1 or a multiple of 4. Using the sets A, B, C, D defined in Lemma 2.3, we can give a more detailed description of the action of f.

Lemma 4.1. *Let $(X, \{P, Q\})$ be a 2–poset with isomorphism $f : (X, P) \to (X, Q)$. If $xP(xf)$, $xP(xf^2)$ and $(xf^2)P(xf^3)$ then (xf^2) $P(xf^i)$ for $i \geq 2$.*

PROOF. We prove this by induction on i. It is clear for $i = 2$ and $i = 3$. Suppose that $(xf^2)P(xf^{2r})$ and $(xf^2)P(xf^{2r+1})$. We have to show that $(xf^2)P(xf^{2r+2})$ and $(xf^2)P(xf^{2r+3})$.

By transitivity, we have $xP(xf^{2r})$ and $xP(xf^{2r+1})$. Therefore xf^2 is P–related to both xf^{2r+2} and xf^{2r+3}. If $(xf^{2r+2})P(xf^2)$ then by transitivity, we get $(xf^{2r+2})P(xf^3)$, which is a contradiction to $(xf^2)P(xf^{2r+1})$, hence $(xf^2)P(xf^{2r+2})$. It follows that $(xf^3)Q(xf^{2r+3})$.
If $(xf^{2r+3})P(xf^2)$ then we get $(xf^{2r+3})P(xf^3)$ by transitivity, which is contradiction again. Therefore $(xf^2)P(xf^{2r+3})$, which concludes the induction and the proof. □

Lemma 4.2. *Let $(X, \{P, Q\})$ be a 2–poset with isomorphism $f : (X, P) \to (X, Q)$. If $xP(xf)$, $(xf^2)P(xf^3)$ and $xQ(xf^2)$ then for all $i \leq 0$ we have $(xf^i)P(xf)$.*

PROOF. First note that from $xQ(xf^2)$ it follows that $(xf^{-1})P(xf)$. The elements xf and xf^3 must be P–related. If $(xf^3)P(xf)$ then by transitivity, we have $(xf^2)P(xf)$ which is a contradiction to $(xf)Q(xf^2)$. Therefore $(xf)P(xf^3)$. We prove the lemma by induction on i. It is clear for $i = 0$ and $i = -1$. Suppose we have $(xf^{-2r})P(xf)$ and $(xf^{-2r-1})P(xf)$ for some $r \geq 0$. We have to show that $(xf^{-2r-2})P(xf)$ and $(xf^{-2r-3})P(xf)$.

By transitivity, we have $(xf^{-2r})P(xf^3)$ and $(xf^{-2r-1})P(xf^3)$, thus both xf^{-2r-2} and xf^{-2r-3} must be P–related to xf. If $(xf)P(xf^{-2r-2})$ then, by transitivity, we get $(xf^{-2r})P(xf^{-2r-2})$, which is a contradiction, as $xQ(xf^2)$, and thus xf^{-2r} and xf^{-2r-2} must be Q–related. Therefore $(xf^{-2r-2})P(xf)$. If $(xf)P(xf^{-2r-3})$ then $(xf^{-2r-2})P(xf^{-2r-3})$ by transitivity, which also gives a contradiction, as xf^{-2r-2} and xf^{-2r-3} must be Q–related, because xf and xf^2 are Q–related. Therefore $(xf^{-2r-3})P(xf)$, which completes the proof. □

Theorem 4.3. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Let A, B, C, D be defined as in Lemma 2.3. Then $Af = B$; $Bf = C$; $Cf = D$; $Df = A$.*

PROOF. Using Lemma 2.3 and counting, it is clear that it is sufficient to prove that $Bf \subseteq C$ and $Df \subseteq A$. Let $y \in B$. Then there exists $x \in A$ with $xf = y$. As X is finite, there exists $n > 0$ minimal with respect to $xf^n = x$. Note that n must be even if we suppose that x is not fixed under f. Now we have $xP(xf)$, and the elements xf^n and xf^{n+1} are P–comparable if and only if n is even. Let us assume that $yf = xf^2 \in A$, which means that $(xf^2)P(xf^3)$. Note that we must have either $xP(xf^2)$ or $xQ(xf^2)$, as $xP(xf)$ and $(xf)Q(xf^2)$. If $xP(xf^2)$, from Lemma 4.1 we get $(xf^2)P(xf^n)$, thus $(xf^2)Px$, and hence $x = xf^2$, thus x is fixed under f^2, and hence also under f. If $xQ(xf^2)$, from Lemma 4.2 we get $(xf^{-n+2})P(xf)$, and thus $(xf^2)P(xf)$, from which it follows that x is fixed under f, because also $(xf)Q(xf^2)$. Therefore, if $yf \in A$ then y is fixed under f, and we also have $yf \in C$. We thus have shown that $Bf \subseteq C$. Dually, we have $Df \subseteq A$. $\square$

Note that Theorem 4.3 is a result which is particular to self–complementary comparability graphs. It is easy to construct examples to show that a similar statement does not hold for self–complementary directed graphs in general.

5. Extensions of 2–posets

We have already seen that if $(X, \{P, Q\})$ is a 2–poset with an isomorphism $f : (X, P) \to (X, Q)$ then f has at most one fixed point. Also, if z is a fixed point of f then $(X \setminus \{z\}, \{P', Q'\})$ is a 2–poset (where P', Q' are the induced orders), and the restriction of f to $X \setminus \{z\}$ gives an isomorphism $(X \setminus \{z\}, P') \to (X \setminus \{z\}, Q')$. In self–complementary graphs, there are, in general, many ways in which this process can be reversed. We shall see how this works for 2–posets. In the following, we shall take A, B, C, D to be defined as in Lemma 2.3, and $A' = A$ if f has no fixed point and $A' = A \setminus \{z\}$ if f has the fixed point z, similarly B', C', D'.

Lemma 5.1. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Let $a \in A'$, $b \in B'$, $c \in C'$, $d \in D'$. Then we can not have any of the following: bPa, cQb, cPd, dQa, cPa, cQa, bPd, dQb. Furthermore, if f has a fixed point z, we can not have any of zPa, zQa, cPz, cQz, bPz, zQb, zPd, dQz.*

PROOF. If bPa then $(bf)Q(af)$. As we also have $aP(af)$ and $bQ(bf)$, we get $bP(af)$ and $bQ(af)$, which gives a contradiction. In the other cases, contradictions can be obtained in similar ways. $\square$

Lemma 5.2. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$ which has a fixed point z. Let $x \in X \setminus \{z\}$. Then the following are equivalent.*

(1) x *and* xf^2 *are* P*–comparable.*

(2) *x and z are P–comparable.*
(3) *z and xf^2 are P–comparable.*

PROOF. Let $x \in A'$. By Theorem 4.3, we then have $xf^2 \in C'$. Suppose (1) holds. By Lemma 5.1 we get $xP(xf^2)$, and we also must have xPz or xQz. If xQz then xf^2 and z are Q–comparable, and as $xf^2 \in C'$ by Lemma 5.1 we have $zQ(xf^2)$. Hence $xQ(xf^2)$, which is a contradiction. Therefore (2) holds. The fact that (2) implies (3) is trivial. Next suppose (3) holds. Using Lemma 5.1, we get $zP(xf^2)$, hence x and z are P–comparable, therefore xPz, and thus $xP(xf^2)$, giving (1). For $x \in C'$ the proof is dual. Note that then for $x \in A'$ or $x \in C'$ we also have the corresponding equivalences with Q–comparability. For $x \in B'$ and $x \in D'$ the results then follow by application of f and using the results for A' and C'. $\square$

Lemma 5.3. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Let x, y be in the same set of A', B', C' and D'. If x, xf^2 are P–comparable and y, yf^2 are P–comparable then x, yf^2 are P–comparable. If x, xf^2 are Q–comparable and y, yf^2 are Q–comparable then x, yf^2 are Q–comparable.*

PROOF. Let $x, y \in A'$, and suppose that x, xf^2 are P–comparable and y, yf^2 are P–comparable. By Lemma 5.1, we have $xP(xf^2)$ and $yP(yf^2)$. It then follows that xf^2 and xf^4 are P–comparable, and as $xf^2 \in C'$ and $xf^4 \in A'$ by Theorem 4.3, we get with Lemma 5.1 that $(xf^4)P(xf^2)$. Similarly, we have $(yf^4)P(yf^2)$. Now by Lemma 5.1 we get $xP(yf^2)$ or $xQ(yf^2)$. Suppose $xQ(yf^2)$. Then xf^2 and yf^4 are Q– comparable, hence we must have $(yf^4)Q(xf^2)$. But now consider xf^2 and yf^2. If $(xf^2)P(yf^2)$ then $xP(yf^2)$; if $(xf^2)Q(yf^2)$ then $(yf^4)Q(yf^2)$; if $(yf^2)P(xf^2)$ then $(yf^4)P(xf^2)$, and if $(yf^2)Q(xf^2)$ then $xQ(xf^2)$. Hence in any case we get a contradiction. Therefore we must have $xP(yf^2)$. The other cases follow similarly. $\square$

Lemma 5.4. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Then we have the following.*
(a) *If $a \in A'$, $b \in B'$ such that $aP(af^2)$ and $(bf^2)Pb$ then aPb.*
(b) *If $c \in C'$, $d \in D'$ such that $(cf^2)Pc$ and $dP(df^2)$ then dPc.*
(c) *If $a \in A'$, $d \in D'$ such that $aQ(af^2)$ and $(df^2)Qd$ then aQd.*
(d) *If $b \in B'$, $c \in C'$ such that $bQ(bf^2)$ and $(cf^2)Qc$ then bQc.*

PROOF. We first prove (a). Note that we have $aP(af)$, $(af)Q(af^2)$, $(af^3)P(af^2)$, $bQ(bf)$, $(bf^2)P(bf)$. By Lemma 5.1, we can not have bPa. Next suppose that aQb. We can have neither $bQ(af^2)$ (as then $aQ(af^2)$), nor $(af^2)Pb$ (as then aPb), nor $bP(af^2)$, as then $(bf^2)P(af^2)$, but af^2

and bf^2 must be Q–comparable. Thus we have $(af^2)Qb$ in contradiction to Lemma 5.1 (as $af^2 \in C$, $b \in B$). Therefore we can not have aQb.

Then let us suppose that bQa. We can have neither $(af^2)Pb$ (as then aPb), nor $(af^2)Qb$ (as $af^2 \in C$, $b \in B$), nor $bP(af^2)$ (as then $(bf^2)P(af^2)$, but af^2 and bf^2 must be Q–comparable). Hence we have $bQ(af^2)$, and thus $(af^3)P(bf)$, we also get $(bf^2)Q(af^2)$ (if we had $(af^2)Q(bf^2)$ then $bQ(bf^2)$), and hence $(bf)P(af)$. Now consider bf and af^2. We can have neither $(bf)P(af^2)$ (as then $(bf^2)P(af^2)$), nor $(af^2)Q(bf)$ (as then $(bf^2)Q(bf)$), nor $(af^2)P(bf)$ (as then $(af^3)Q(bf^2)$, and hence $(af^3)Q(af^2)$). Therefore we must have $(bf)Q(af^2)$. As $(bf)P(af)$, we also get $(bf^2)P(af)$, hence a and bf must be Q–comparable. Thus we have $aQ(bf)$, but then $aQ(af^2)$, which gives the final contradiction. Therefore we must have aPb.

The assertion (b) follows from (a) by duality, and furthermore (c) and (d) follow from (a) and (b) by application of f. □

Theorem 5.5. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. If f has a fixed point z then $(X \setminus \{z\}, \{P', Q'\})$ is a 2–poset (where P' and Q' are the induced orders), and the restriction of f to $X \setminus \{z\}$ is an isomorphism $(X \setminus \{z\}, P') \to (X \setminus \{z\}, Q')$. Conversely, if f does not have a fixed point, and $z \notin X$ then there exist unique partial orders $\bar{P}$ and $\bar{Q}$ on $\bar{X} = X \cup \{z\}$ such that P and Q are induced by $\bar{P}$ and $\bar{Q}$ respectively, such that $(\bar{X}, \{\bar{P}, \bar{Q}\})$ is a 2–poset and $\bar{f} : \bar{X} \to \bar{X}$ defined by $z\bar{f} = z$ and $x\bar{f} = xf$ for $x \in X$ is an isomorphism $(\bar{X}, \bar{P}) \to (\bar{X}, \bar{Q})$.*

PROOF. If f has a fixed point then the assertion is obvious. Let us assume that f has no fixed point. From Lemma 5.1 and Lemma 5.2 it follows that there is at most one way of defining partial orders $\bar{P}, \bar{Q}$ on $\bar{X}$ with the required conditions (for example, if $x \in A$ then xPz if x and xf^2 are P–comparable and xQz if x and xf^2 are Q–comparable). On the other hand, if we define relations $\bar{P}, \bar{Q}$ in this way, then from Lemmas 5.1 – 5.4 it follows that the relations are partial orders (one has to show transitivity via z), and that $\bar{f}$ is an isomorphism. □

We can also construct extensions by more than one element.

Theorem 5.6. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Let $\{a, b, c, d\}$ be a 4–element set disjoint from X. Then on $\bar{X} = X \cup \{a, b, c, d\}$ there exist partial orders $\bar{P}, \bar{Q}$ such that P and Q are induced by $\bar{P}$ and $\bar{Q}$ respectively, such that $(\bar{X}, \{\bar{P}, \bar{Q}\})$ is a 2–poset, and there exists an isomorphism $\bar{f} : (\bar{X}, \bar{P}) \to (\bar{X}, \bar{Q})$ with $x\bar{f} = xf$ for all $x \in X$.*

PROOF. First assume that f has no fixed point. Define all relations between elements $x \in X$ and a, b, c, d the same as the relations in the construction of the fixed point in Theorem 5.5. Furthermore, define aPb, aPc, aQd, bQc, bQd, dPc. Define $\bar{f}$ by $x\bar{f} = xf$ for all $x \in X$ and $af = b$, $bf = c$, $cf = d$ and $df = a$. Thus using Lemmas 5.1 – 5.4 as in Theorem 5.5 we get the result. If f has a fixed point, then we can remove it, add the points a, b, c, d as above, and then add the fixed point again by Theorem 5.5. It is clear that the relations between the fixed point and the other elements of X are the same as before, which concludes the proof of the theorem. $\square$

6. Extremal elements

We finally want to identify some elements of A, B, C and D. First note that for any 2–poset $(X, \{P, Q\})$ we can define linear orders T and S on X by xTy if xPy or xQy, and xSy if xPy or yQx. It is now natural to look at the maximal and minimal elements of these orders (which exist, if X is finite). If (X, P) is a poset and $x \in X$ we say that x is P–minimal (P–maximal) if x is a minimal (maximal) element of (X, P). Note that by what we said above in a finite 2–poset $(X, \{P, Q\})$ there exists a unique element which is P–minimal and Q–minimal, and similarly for the other combinations.

Proposition 6.1. *Let $(X, \{P, Q\})$ be a finite 2–poset with isomorphism $f : (X, P) \to (X, Q)$. Let A, B, C, D be defined as in Lemma 2.3, let a be P–minimal and Q–minimal, let b be P–maximal and Q–minimal, let c be P–maximal and Q–maximal, and let d be P–minimal and Q–maximal. Then $a \in A$, $b \in B$, $c \in C$, $d \in D$.*

PROOF. By minimality of a, we must have $aP(af)$ or $aQ(af)$. If $aQ(af)$ then $(af^{-1})Pa$ in contradiction to P–minimility of a, hence $aP(af)$, and $a \in A$. Similarly, we must have $(bf)Pb$ or $bQ(bf)$. But if $(bf)Pb$ then $b \in C$, and $bf^{-1} \in B$, hence $(bf^{-1})Qb$ in contradiction to Q–minimality of b. Therefore $bQ(bf)$ and $b \in B$. By duality, we get $c \in C$ and $d \in D$. $\square$

Proposition 6.2. *Let $X, P, Q, f, A, B, C, D, a, b, c, d$ be as in Proposition 6.1. Suppose $|X| > 5$, and let a' be P'–minimal and Q'–minimal in $X' = X \setminus \{a, b, c, d\}$, where P' and Q' are induced by P and Q respectively, and let b', c', d' be defined similarly. Then $a' \in A$, $b' \in B$, $c' \in C$, $d' \in D$.*

PROOF. First consider the different possibilities for a'. First suppose $a' \in C$. Choose $x \in A \setminus \{a\}$. By Lemma 5.1, we must have xPa' or xQa', which is a contradiction to minimality of a'. Next suppose $a' \in D$. Then $(a'f)Qa'$, thus by minimality of a', we have $a'f = a$. We also have aQd, and, by minimality of a', we must have $a'P(df)$. Now we can not have dPa', as then $(df)Qa$, and hence $df = a$, which is a contradiction. Thus we get $a'Qd$, and we must have $aP(df)$. But from $a'P(df)$ it follows that $aQ(df^2)$, but $aP(df)$ and $(df)P(df^2)$, hence $aP(df^2)$, which is a contradiction. Next we suppose that $a' \in B$. As then $(a'f^{-1})Pa'$, we must have $a'f^{-1} = a$, and by minimality of a', we get $a'Q(bf')$. As $(bf^{-1})Pb$ we can not have bQa' (as then would follow $bQ(bf^{-1})$), therefore we get $a'Pb$, and hence $aQ(bf^{-1})$. From $a'Q(bf^{-1})$ it follows that $aP(bf^{-2})$. But as $(bf^{-1})Q(bf^{-2})$, we also have $aQ(bf^{-2})$, which is a contradiction, thus we are only left with $a' \in A$.

Next consider the possibilities for b'. First suppose $b' \in D$. We have $(b'f)Qb'$ and $b'P(b'f^{-1})$. By P–maximality of b', we have $c = b'f^{-1}$, and by Q–minimality of b', we get $b'f = a$. We also have aQb and dPc. If dPb' then $(df)Q(b'f)$, hence $(df)Qb'$, giving a contradiction. We thus have $b'Qd$. But then $(b'f^{-1})P(df^{-1})$ and $b'P(df^{-1})$, giving also a contradiction. Next suppose $b' \in A$. Then $b'P(b'f)$, and hence $b'f = b$. We then must have $b'Q(af)$, and as $aP(af)$, we get aPb', hence $(af)Qb$. But then $b'Qb$, which is a contradiction. Now suppose $b' \in C$. We then have $(b'f^{-1})Qb'$, and hence $b'f^{-1} = b$. We then must have $(cf^{-1})Pb'$. Now if $b'Qc$ then $bP(cf^{-1})$, and hence bPb', giving a contradiction, therefore $b'Pc$. But then also $(cf^{-1})Pc$, which is a contradiction as we must have $(cf^{-1})Qc$. Therefore we are left with $b' \in B$. Finally by duality, we get $c' \in C$ and $d' \in D$. □

We finally remark that although $a \in A$, $b \in B$, $c \in C$, $d \in D$, these extremal elements do not seem to be connected with the isomorphism f in any closer way. For example, the set $\{a, b, c, d\}$ does not need to be an orbit of f, in fact, it is not hard to construct a 2–poset $(X, \{P, Q\})$ with isomorphism $f : (X, P) \to (X, Q)$ such that $\{a, b, c, d\}$ is not an orbit of any isomorphism $g : (X, P) \to (X, Q)$.

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