Publ. Math. Debrecen
38 / 3–4 (1991), 231–235

Bounding functions concerning character degrees of solvable groups

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To the memory of Professor Béla Barna

Throughout the paper G always denotes a solvable group of finite order. We shall use the notations of [6]. We shall investigate some properties of the functions $\alpha(r)$ and $\delta(r)$ of [3] and [5]:

$$\alpha(r) = \max\{d.l.\,(G/\operatorname{Ker}\chi) \mid \chi \in \operatorname{Irr}(G), \quad \chi(1) \leq f_r\}$$

where

$$c.d.(G) = \{f_1 < f_2 \cdots < f_n\}$$

and

$$\delta(1) = 1 \quad , \quad \delta(r) = \alpha(r) - \alpha(r-1) \quad \text{for } r = 2, \ldots, n.$$

We know from [5] that $\delta(i) \leq 3$ for $i = 1, \ldots, n$ and according to [3] $\alpha(i) \leq 2i$ for $i = 1, \ldots n$. Our aim is to find properties which enable us to say somewhat more about these functions.

Let us introduce the following

Hypothesis 1. Let $\psi \in \operatorname{Irr}(G)$ and $M \triangleleft G$. If $M \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ such that $\chi(1) < \psi(1)$ then $M' \leq \operatorname{Ker}\psi$.

Hypothesis 1'. The same as Hypothesis 1 but M must be a member of the derived series of G.

Remark 1. According to [1], all solvable groups of odd order satisfy Hypoyhesis 1. We can see easily that M–groups also satisfy it since if $\psi \in \operatorname{Irr}(G)$ then $\psi = \lambda^G$ for some $\lambda \in \operatorname{Irr}(H)$ linear character of $H \leq G$. If ψ is not linear then every nonprincipal irreducible constituent of 1_H^G has degree less than $\psi(1)$. So $M \leq \operatorname{Ker} 1_H^G \leq H$. But then $M' \leq H' \leq \operatorname{Ker}\lambda$ and so also $M' \leq \operatorname{Ker}\psi$. It is trivial that Hypothesis 1 yields Hypothesis 1'.

First we prove the following simple

Lemma 1. *If the group G satisfies Hypothesis 1' then $\delta(i) \leq 1$ for $i = 1, \dots, n$ and consequently $\alpha(i) \leq i$ for $i = 1, \dots, n$. Conversely, if $\delta(i) \leq 1$ for $i = 1, \dots, n$ then Hypothesis 1' is satisfied in G.*

PROOF. We use induction. $\delta(1) = 1$ by definition. Let us suppose that Hypothesis 1' is satisfied in G and from this we have already proved that $\delta(i) \leq 1$ for $i = 1, \dots, r-1$. By the definition of the function α , $G^{\alpha(r-1)} \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ satisfing $\chi(1) \leq f_{r-1}$. From Hypothesis 1' we have that $G^{\alpha(r-1)+1} \leq \operatorname{Ker}\psi$ for all $\psi \in \operatorname{Irr}(G)$ satisfying $\psi(1) \leq f_r$. So $\alpha(r) \leq \alpha(r-1) + 1$ that is $\delta(r) \leq 1$. As $\alpha(r) = \sum_1^r \delta(i) \leq r$, the first part of the statement is proved.

Conversely, if $\delta(i) \leq 1$ for $i = 2, \dots, n$ then $\alpha(i) \leq \alpha(i-1) + 1$ so $G^{\alpha(i-1)+1} \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ satisfying $\chi(1) \leq f_i$ and from this Hypothesis 1' easily follows.

Corollary 1. *Groups of odd order and M–groups satisfy $\delta(i) \leq 1$ and $\alpha(i) \leq i$ for $i = 1, \dots, n$.*

PROOF. Follows from Remark 1 and Lemma 1.

Now we weaken our hypotheses:

Hypothesis 2. Let $M \triangleleft G$ and let us suppose that $M \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ satisfying $\chi(1) \leq f_{r-1}$. If there exists a $\psi \in \operatorname{Irr}(G)$ for which $M' \not\leq \operatorname{Ker}\psi$ and $\psi(1) = f_r$ then $M'' \leq \operatorname{Ker}\vartheta$ for all $\vartheta \in \operatorname{Irr}(G)$ satisfying $\vartheta(1) \leq f_{r+1}$. Here $r \leq n$ and we define f_{n+1} to be equal to f_n.

Hypothesis 2'. The same as Hypothesis 2 but M must be a member of the derived series of G.

Before dealing with Hypothesis 2 or Hypothesis 2' we introduce a further weakening of them, namely

Hypothesis 2.* Let $M \triangleleft G$ and $\chi \in \operatorname{Irr}(G)$. Let us suppose that $M \leq \operatorname{Ker}\varphi$ for all $\varphi \in \operatorname{Irr}(G)$ such that $\varphi(1) < \chi(1)$; then $M'' \leq \operatorname{Ker}\chi$.

Hypothesis 2'.* The same as Hypothesis 2* but M must be a member of the derived series of G.

Remark 2. It is easy to see that Hypothesis 1 yields Hypothesis 2 and Hypothesis 2 yields Hypothesis 2*. The same is true for Hypothesis i' for i $= 1, 2, 2^*$.

As we can see e.g. in the case of SL(2,3) Hypothesis 2 does not yield Hypothesis 1'. Here $\alpha(1) = 1,\ \alpha(2) = 3,\ \alpha(3) = 3$ and $\delta(1) = 1,\ \delta(2) = 2$ and $\delta(3) = 0$.

Now we have a similar statement as in Lemma 1:

Lemma 2. *Hypothesis 2*' is equivalent to $\delta(i) \leq 2$ for $i = 1, \ldots, n$.*

PROOF. Easy to prove.

For Hypothesis 2' we can prove more:

Lemma 3. *If G satisfies Hypothesis 2' then $\delta(i) \leq 2$ for $i = 1, \ldots, n$ and if for some $r < n$ $\delta(r) = 2$ then $\delta(r+1) = 0$.*

PROOF. The first statement follows from Lemma 2. If $\delta(r) = 2$ for some $r < n$ then by Hypothesis 2' $G^{\alpha(r-1)+2} \leq \operatorname{Ker}\vartheta$ for every $\vartheta \in \operatorname{Irr}(G)$ satisfying $\vartheta(1) \leq f_{r+1}$. So we have that $\alpha(r+1) \leq \alpha(r-1) + 2$ that is $\delta(r+1) + \delta(r) \leq 2$. So $\delta(r+1) = 0$.

In a similar way as in the proof of Corollary 9 of [3] we get the following

Theorem 1. *If G satisfies Hypothesis 2' then $\alpha(r) \leq r + 1$ for all $r \leq n$. If for an r $\delta(r) \leq 1$ is also true then $\alpha(r) \leq r$.*

PROOF. By Lemma 3 we have that $\delta(i) \leq 2$ for $i = 1, \ldots, n$. If $\delta(i) \leq 1$ for all $i = 1, \ldots, n$ then we are done by Lemma 1. If $\delta(i) = 2$ for some $i < n$ then by Lemma 3 we have that $\delta(i+1) = 0$. Set

$$S = \{i \mid i < r, \ \ \delta(i) = 2\},$$
$$T = \{i \mid i \leq r, \ \ i - 1 \in S\}.$$

Then $T \cap S = \emptyset$ and $|S| = |T|$.
So $\alpha(r) = \sum_1^r \delta(i) \leq 1 + \delta(r) + 2|S| + 0 \cdot |T| + 1 \cdot (r - |T \cup S \cup \{1, r\}|) =$

$$= \begin{cases} r & \text{if } \ \delta(r) = 0 \text{ and } r \in T \text{ or } \delta(r) = 1 \\ r - 1 & \text{if } \ \delta(r) = 0 \text{ and } r \notin T \\ r + 1 & \text{if } \ \delta(r) = 2. \end{cases}$$

Corollary 2. *If a group G satisfies Hypothesis 2' then $d.l.(G) \leq |c.d.(G)| + 1$. If in addition $\delta(n) \leq 1$ then $d.l.(G) \leq |c.d.(G)|$.*

Corollary 3. *If for a group G $d.l.(G) \leq 3$ then $\alpha(i) \leq i + 1$ for $i = 1, \ldots, n$. This bound is strict. These groups satisfy $\delta(n) \leq 1$ and $d.l.(G) \leq |c.d.(G)|$.*

PROOF. As for these groups Hypothesis 2' is easily seen to be satisfied, the first statement follows from Theorem 1. $\alpha(i) = i + 1$ for $i = 2$ in $SL(2,3)$. It is easy to see that $\delta(n) \leq 1$ so the last assertion also follows from Theorem 1.

Remark 3. Although we cannot prove in general that Hypothesis 2' yields $d.l.(G) \leq |c.d.(G)|$, we mention that it is true e.g. in the case of Frobenius groups. Frobenius groups satisfy Hypothesis 2' if and only if their complement H satisfies it and $\delta_H(|c.d.(H)|) \leq 1$.

Let us introduce for $\psi \in \mathrm{Irr}(G)$

Hypothesis 2 (ψ). Let $\psi(1) = f_r$. Let $M \triangleleft G$ such that $M \leq \operatorname{Ker}\chi$ for all $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \leq f_{r-1}$. If $M' \not\leq \operatorname{Ker}\psi$ then $M'' \leq \operatorname{Ker}\vartheta$ for all $\vartheta \in \mathrm{Irr}(G)$ such that $\vartheta(1) \leq f_{r+1}$. Here f_{n+1} is defined to be equal to f_n.

Using Lemma 7 of [3] we can prove the following

Lemma 4. *If G is a finite solvable group such that $3, 5 \nmid |G|$ and $\psi \in \mathrm{Irr}(G)$ is faithful and primitive then Hypothesis 2 (ψ) is satisfied in G.*

PROOF. Let $M \triangleleft G$ such that $M \triangleleft G$ such that $M \leq \operatorname{Ker}\chi$ for all $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) < \psi(1)$. Let us assume that $M'' \neq 1$. Then $[M, M'] \neq 1$ either.

By Lemma 7 of [3] there exists an $\omega \in \mathrm{Irr}(G)$ such that $\psi(1) < \omega(1) \leq 3/2\,\psi(1)$ and $M \not\leq \operatorname{Ker}\omega$. Also $\psi(1) = 2$ or $\psi(1) = 4$. So either $2 < \omega(1) \leq 3/2{\cdot}2 = 3$, which yields that $\omega(1) = 3$, or $4 < \omega(1) \leq 3/2{\cdot}4 = 6$, which yields $\omega(1) = 5$ or 6. As $\omega(1) \mid |G|$, these are all impossible. So $M'' = 1$ and we are done.

Remark 4. Hypothesis 2*' does not hold for all solvable groups as we can see in the example of $GL(2,3)$. Here we have $\alpha(1) = 1,\ \alpha(2) = 4,$ $\alpha(3) = 4,\ \alpha(4) = 4$ and $\delta(1) = 1,\ \delta(2) = 3,\ \delta(3) = 0,\ \delta(4) = 0$. So for $i = 2$ we have $\alpha(i) = i + 2$ and $\delta(i) = 3$.

Now we prove a sufficient condition for Hypothesis 2*. This can be considered as an extension of part b/ of Theorem 6 in [5].

Theorem 2. *Let G be a finite solvable group such that $3, 5 \nmid |G|$ then Hypothesis 2 * is satisfied in G.*

PROOF. Let G be a counterexample of minimal order. Let $M \triangleleft G$, $\chi \in \mathrm{Irr}(G)$; we suppose that $M'' \not\leq \operatorname{Ker}\chi$ and $M \leq \operatorname{Ker}\varphi$ for all $\varphi \in \mathrm{Irr}(G)$ with $\varphi(1) < \chi(1)$. By Lemma 4, χ cannot be faithful and primitive.

If $\operatorname{Ker}\chi > 1$ then by the inductive hypothesis applied to χ, $M \operatorname{Ker}\chi / \operatorname{Ker}\chi$ and $G/\operatorname{Ker}\chi$ $\ M'' \leq \operatorname{Ker}\chi$, which is a contradiction.

If χ is not primitive then $\chi = \alpha^G$ for an $\alpha \in \mathrm{Irr}(H)$ for some $H < G$. As $1_H^G \notin \mathrm{Irr}(G)$, $M \leq \operatorname{Ker} 1_H^G \leq H$. If $\nu \in \mathrm{Irr}(H)$ such that $\nu(1) < \alpha(1)$ then as $\nu^G(1) < \chi(1)$ we have $M \leq \operatorname{Ker}\nu^G \leq \operatorname{Ker}\nu$ and by the inductive hipothesis applied to M, H and α we get $M'' \leq \operatorname{Ker}\alpha$ so $M'' \leq \operatorname{Ker}\chi$, and thus we have a final contradiction.

In [4] we have proved that Hypothesis 1 is inherited by direct products. Similarly, one can prove easily that this is also true for Hypothesis $2^\star$, namely we have the following

Lemma 5. *Let $G = G_1 \times G_2$ where G_i $i = 1, 2$ satisfy Hypothesis $2^\star$. Then G satisfies it as well.*

Now we have an analogous statement as Theorem 1.5 in [4]:

Theorem 3. *Let G be a finite solvable group such that for every $x \in G^\star$, $C_G(x)$ is the direct product of a $\{3,5\}$-group and of a $\{3,5\}'$ group. Then G satisfies Hypothesis $2^\star$.*

PROOF. Using the above Remark 1 and Lemma 5 with Theorem B and Lemma 2 of [2] the proof is similar to that of Theorem 1.5 in [4].

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Acknowledgements

The author would like to express her gratitude to Professor HUPPERT for drawing her attention to Gluck's paper and also for his comments on the previous version of this paper. Consultations with Professor K. CORRÁDI and P. HERMANN were also very helpful.

Research partially supported by Hungarian National Foundation for Scientific Research grant no. 1813.

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(Received April 14, 1988)