

New characterizations of the arithmetic-geometric mean of Gauss and other well-known mean values

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Dedicated to Professor Z. Daróczy on his 50th birthday

1. Introduction and statement of the results

Throughout this paper, a mean value of two positive real numbers a, b denoted by $M(a, b)$, is defined to be a real number which satisfies the following postulates:

- $(P_1) \quad M : R^+ \times R^+ \rightarrow R;$
- $(P_2) \quad M(a, b) = M(b, a) \quad (\text{symmetry property});$
- $(P_3) \quad M(a, a) = a \quad (\text{reflexivity property}).$

See [2].

We consider the complete elliptic integral of the first kind

$$(1) \quad I(a, b) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}},$$

where a, b are arbitrary positive real numbers. Throughout this paper a, b denote arbitrary positive real numbers.

The above $I(a, b)$ is closely related to the arithmetic-geometric mean of Gauss of a, b denoted by $G(a, b)$ (see [5], [7] and [9]). Indeed, the following equality holds in $R^+ \times R^+$:

$$(2) \quad G(a, b) = \frac{1}{I(a, b)}.$$

It is well-known that $f(a, b) = G(a, b)$ satisfies the following functional equation in $R^+ \times R^+$:

$$(3) \quad f\left(\frac{a+b}{2}, \sqrt{ab}\right) = f(a, b),$$

where $f : R^+ \times R^+ \rightarrow R$ and f is an unknown function. Hence, by (2) $f(a, b) = I(a, b)$ satisfies (3). Throughout this paper, for the sake of simplicity, we denote $\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$ by r .

The purpose of Section 3 is to prove the following theorem:

Theorem 1. *Let $f : R^+ \times R^+ \rightarrow R$. If f can be represented by the form, containing some function p , in $R^+ \times R^+$*

$$(4) \quad f(a, b) = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta,$$

where $p : R^+ \rightarrow R$ and $p''(x)$ is continuous in R^+ , then the only solution of (3) is given by

$$f(a, b) = A I(a, b) + B = A \frac{1}{G(a, b)} + B,$$

where A, B are arbitrary real constants.

Before proceeding to Theorem 2 we require the additional condition on p that p be strictly monotonic in R^+ . We set (see [4])

$$(5) \quad M(a, b; p) \stackrel{\text{def}}{=} p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta \right).$$

Throughout this paper, for the sake of simplicity, we write $M(a, b)$ for $M(a, b; p)$, i.e.,

$$M(a, b) \stackrel{\text{def}}{=} p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta \right).$$

If we set $p(r) = \frac{1}{r}$, $p(r) = \log r$, $p(r) = \frac{1}{r^2}$, $p(r) = r^2$ in (5), by using (1), (2) and the three definite integrals

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta &= \log \frac{a+b}{2}, \\ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} &= \frac{1}{ab}, \\ \frac{1}{2\pi} \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta &= \frac{a^2 + b^2}{2},\end{aligned}$$

we obtain $M(a, b) = G(a, b)$, $M(a, b) = \frac{a+b}{2}$, $M(a, b) = \sqrt{ab}$,
 $M(a, b) = \sqrt{\frac{a^2+b^2}{2}}$, respectively.

The purpose of Section 4 is to prove the following theorem:

Theorem 2.

- (i) $M(a, b) = G(a, b)$ holds for all positive real numbers a, b iff $p(r) = A\frac{1}{r} + B$ where $A(\neq 0)$, B are arbitrary real constants.
- (ii) $M(a, b) = \frac{a+b}{2}$ holds for all positive real numbers a, b iff $p(r) = A \log r + B$ where $A(\neq 0)$, B are arbitrary real constants.
- (iii) $M(a, b) = \sqrt{ab}$ holds for all positive real numbers a, b iff $p(r) = A\frac{1}{r^2} + B$ where $A(\neq 0)$, B are arbitrary real constants.
- (iv) $M(a, b) = \sqrt{\frac{a^2+b^2}{2}}$ (the root-mean-square of a, b) holds for all positive real numbers a, b iff $p(r) = Ar^2 + B$ where $A(\neq 0)$, B are arbitrary real constants.
- (v) There exists no $p(r)$ such that $M(a, b) = \frac{2ab}{a+b}$ (the harmonic mean of a, b) holds for all positive real numbers a, b .

To prove Theorem 1 and Theorem 2 the lemma in Section 2 plays an important part.

Remark. About means see [1], pp. 234–244, [2]–[8] and [10]–[13].

2. Lemma

Lemma. Let $p : R^+ \rightarrow R$. We assume that $p''(x)$ is continuous in R^+ . If we set

$$(6) \quad f(a, b) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} p\left(\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}\right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta,$$

then

$$\begin{aligned} (i) \quad f_a(c, c) &= f_b(c, c) = \frac{1}{2}p'(c), \\ (ii) \quad f_{aa}(c, c) &= f_{bb}(c, c) = \frac{3cp''(c) + p'(c)}{8c}; \\ (iii) \quad f_{ab}(c, c) &= f_{ba}(c, c) = \frac{cp''(c) - p'(c)}{8c}, \end{aligned}$$

where c is an arbitrary positive real number.

PROOF. Throughout the proof we apply differentiation under the integral sign which can be done because $p(r)$ is of class C^2 in $R^+ \times R^+$ with respect to a, b .

Proof of (i). By (6) we obtain

$$(7) \quad f_a(a, b) = \frac{1}{2\pi} \int_0^{2\pi} p'(r) \frac{a}{r} \cos^2 \theta \, d\theta.$$

Setting $a = b = c$ in (7) and using $\sqrt{c^2(\cos^2 \theta + \sin^2 \theta)} = c$ yields

$$\begin{aligned} f_a(c, c) &= \frac{1}{2\pi} \int_0^{2\pi} p'(c) \frac{c}{c} \cos^2 \theta \, d\theta = \frac{1}{2\pi} p'(c) \int_0^{2\pi} \cos^2 \theta \, d\theta = \\ &= \frac{1}{2\pi} p'(c) \pi = \frac{1}{2} p'(c). \end{aligned}$$

Similarly we can prove that

$$f_b(c, c) = \frac{1}{2} p'(c).$$

Proof of (ii). By (7) we obtain

$$f_{aa}(a, b) = \frac{1}{2\pi} \int_0^{2\pi} \left(p''(r) \frac{a^2}{r^2} \cos^2 \theta + p'(r) \frac{r^2 - a^2 \cos^2 \theta}{r^3} \right) \cos^2 \theta \, d\theta.$$

Setting $a = b = c$ in the above equality yields

$$\begin{aligned}
 f_{aa}(c, c) &= \frac{1}{2\pi} \left(p''(c) \int_0^{2\pi} \cos^4 \theta d\theta + \frac{p'(c)}{c} \int_0^{2\pi} \sin^2 \theta \cos^2 \theta d\theta \right) = \\
 &= \frac{1}{2\pi} \left(p''(c) \frac{3\pi}{4} + \frac{p'(c)}{c} \frac{\pi}{4} \right) = \\
 &= \frac{3cp''(c) + p'(c)}{8c}.
 \end{aligned}$$

Similarly we can prove that

$$f_{bb}(c, c) = \frac{3cp''(c) + p'(c)}{8c}.$$

Proof of (iii). We can prove (iii) in a similar manner to that of (ii).
Q.E.D.

3. Proof of Theorem 1

Applying $\frac{\partial^2}{\partial a^2}$ to both sides of (3), using the Chain Rule for differentiation for two real variables and observing that $f_{ab}(a, b) = f_{ba}(a, b)$ yields

$$\begin{aligned}
 (8) \quad & \frac{1}{4} f_{aa} \left(\frac{a+b}{2}, \sqrt{ab} \right) + \frac{\sqrt{b}}{2\sqrt{a}} f_{ab} \left(\frac{a+b}{2}, \sqrt{ab} \right) + \\
 & + \frac{b}{4a} f_{bb} \left(\frac{a+b}{2}, \sqrt{ab} \right) - \frac{\sqrt{b}}{4a\sqrt{a}} f_b \left(\frac{a+b}{2}, \sqrt{ab} \right) = f_{aa}(a, b).
 \end{aligned}$$

Setting $a = b = c$, where c is an arbitrarily fixed positive real number, in (8) yields

$$(9) \quad \frac{3}{4} f_{aa}(c, c) - \frac{1}{2} f_{ab}(c, c) - \frac{1}{4} f_{bb}(c, c) + \frac{1}{4c} f_b(c, c) = 0.$$

Substituting $f_{aa}(c, c)$, $f_{ab}(c, c)$, $f_{bb}(c, c)$, $f_b(c, c)$ in Lemma in Section 2 into (9) and simplifying the resulting equality yields

$$p''(c) + \frac{2}{c} p'(c) = 0.$$

Since c was an arbitrarily fixed positive real number, we can replace c by a positive real variable x in the above equality. So we obtain in R^+

$$p''(x) + \frac{2}{x}p'(x) = 0.$$

Solving the above differential equation yields in R^+

$$p(x) = A\frac{1}{x} + B,$$

and so

$$(10) \quad p(r) = A\frac{1}{r} + B,$$

where A, B are real constants satisfying $A \neq 0$. Substituting (10) into (4) and using (1) yields in $R^+ \times R^+$

$$(11) \quad f(a, b) = AI(a, b) + B.$$

Direct substitution of (11) shows that (11) is a solution of our original functional equation (3).

4. Proof of Theorem 2

Remark. To prove Theorem 2 (i) we shall apply Theorem 1.

Proof of (i). We have only to prove the "only if" part. By hypothesis

$$(12) \quad M(a, b) = G(a, b)$$

holds in $R^+ \times R^+$. By (5), (12) we obtain in $R^+ \times R^+$

$$(13) \quad p(G(a, b)) = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta.$$

Since $G(a, b)$ is a solution of the functional equation (3), $P(G(a, b))$ is also a solution of (3). Furthermore, (13) holds. Hence, by Theorem 1 we obtain in $R^+ \times R^+$

$$(14) \quad p(G(a, b)) = A I(a, b) + B = A \frac{1}{G(a, b)} + B,$$

where A, B are real constants satisfying $A \neq 0$. If we set $b = a$ in (14), then we obtain in R^+

$$(15) \quad p(G(a, a)) = A \frac{1}{G(a, a)} + B.$$

Since $G(a, a) = a$, by (15) we have in R^+

$$p(a) = A \frac{1}{a} + B,$$

or

$$p(r) = A \frac{1}{r} + B. \quad \text{Q.E.D.}$$

Proof of (ii). We have only to prove the "only if" part. By hypothesis

$$(16) \quad M(a, b) = \frac{a + b}{2}$$

holds in $R^+ \times R^+$. By (5), (16) we obtain in $R^+ \times R^+$

$$(17) \quad p\left(\frac{a + b}{2}\right) = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta.$$

If we set in $R^+ \times R^+$

$$(18) \quad f(a, b) = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta,$$

by (17), (18) we have in $R^+ \times R^+$

$$(19) \quad f(a, b) = p\left(\frac{a + b}{2}\right).$$

Applying $\frac{\partial^2}{\partial a^2}$ to both sides of (19) and setting $a = b = c$, where c is an arbitrarily fixed positive real number in the resulting equality yields

$$(20) \quad f_{aa}(c, c) = \frac{1}{4}p''(c).$$

By (18) and by Lemma (ii) in Section 1 we obtain

$$(21) \quad f_{aa}(c, c) = \frac{3cp''(c) + p'(c)}{8c}.$$

Substituting (21) in (20) and simplifying the resulting equality yields

$$p''(c) + \frac{1}{c}p'(c) = 0,$$

and so we obtain in R^+

$$p''(x) + \frac{1}{x}p'(x) = 0.$$

Solving the above differential equation yields in R^+

$$p(x) = A \log x + B,$$

and so

$$p(r) = A \log r + B,$$

where A, B are real constants satisfying $A \neq 0$.

Q.E.D.

Proof of (iii). Since we can prove (iii) by using a similar argument to that in (ii), we omit the proof.

Proof of (iv). Since we can prove (iv) by using a similar argument to that in (ii), we omit the proof.

Proof of (v). The proof is by contradiction. Assume contrary. Then there exists a $p(r)$ such that

$$(22) \quad M(a, b) = \frac{2ab}{a+b}$$

holds for all positive real numbers a, b . By (5), (22) we obtain in $R^+ \times R^+$

$$(23) \quad p\left(\frac{2ab}{a+b}\right) = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta.$$

Starting with (23) and using a similar argument to that in (ii) yields the differential equation for p in R^+

$$p''(x) + \frac{5}{x}p'(x) = 0.$$

Solving the above differential equation yields in R^+

$$p(x) = A\frac{1}{x^4} + B,$$

and so

$$(24) \quad p(r) = A\frac{1}{r^4} + B = A\frac{1}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^2} + B,$$

where A, B are real constants and $A \neq 0$. Substituting (24) into (5) yields

$$(25) \quad M(a, b) = \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^2} \right)^{-\frac{1}{4}}.$$

Since

$$(26) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^2} = \frac{1}{2ab} \left(\frac{1}{a^2} + \frac{1}{b^2} \right),$$

by (22), (25), (26) we obtain for all positive real numbers

$$\frac{2ab}{a+b} = \left(\frac{1}{2ab} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right)^{-\frac{1}{4}}$$

and so

$$a = b.$$

This is a contradiction.

Q.E.D.

Remark. The mean $M(a, b) = \frac{a^2+b^2}{a+b}$ for all positive real numbers a, b is said to be the antiharmonic mean of a, b . As a generalization of the above mean and the arithmetic mean of a, b we consider the mean $M(a, b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ where n is a positive integer. In a similar way to Theorem 2(v) we can prove the following result: If $n > 1$, there exists no $p(r)$ such that $M(a, b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ holds for all positive real numbers a, b .

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