Publ. Math. Debrecen
39 / 1–2 (1991), 171–179

On second order linear divisibility sequences over algebraic number fields

By K. GYŐRY[1] (Debrecen) and A. PETHŐ[1,2] (Debrecen)

To the memory of B. Barna, K. Buzási, S. Buzási and M. Erdélyi

1. Introduction

Let R be an integral domain which is finitely generated over $\mathbf{Z}$. (Linear) *divisibility sequences* over R are recurrence sequences $\{u_h\}_{h=0}^{\infty}$, $u_h \in R$, $h = 0, 1, \dots$ (that is sequences in R satisfying linear homogeneous recurrence relations with constant coefficients) with the property that whenever $h|k$, then $u_h|u_k$ in R. This notation was introduced by M. HALL [4] who described all second order divisibility sequences over $\mathbf{Z}$, as well as the third order divisibility sequences over $\mathbf{Z}$ having irreducible characteristic polynomials.

Let $d > 1$ be an integer. The recurrence sequence $\{u_h\}_{h=0}^{\infty}$ is called d-(linear) *divisibility sequence* if $u_h|u_{hd}$ in R for $h = 0, 1, 2, \dots$. For $R = \mathbf{Z}$, SOLOMON [6] characterized all 2-divisibility sequences. BÉZIVIN, PETHŐ and VAN DER POORTEN [1] proved for any R, that if $\{u_h\}_{h=0}^{\infty}$ is d-divisible for an integer $d > 1$ then there is a recurrence sequence $\{\bar{u}_h\}_{h=0}^{\infty}$ of the form

$$\bar{u}_h = h^k \prod_i \left(\frac{\alpha_i^h - \beta_i^h}{\alpha_i - \beta_i} \right)$$

with some integer $k \geq 0$ over R such that $u_h|\bar{u}_h$ in R for $h = 0, 1, 2, \dots$. Thus they confirmed an old conjecture of WARD [7]. For further references concerning divisibility sequences, we refer to [1].

Although the result of Bézivin et al. is very general, it is not straightforward to deduce from it a complet list of d-divisibility sequences over a

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[1] Research supported in part by Grant 273 from the Hungarian National Foundation for Scientific Research.

[2] Research partly done while the second author was a Visiting Professor at the University of Saarbrücken.

given ring. The aim of this paper is to give a more explicit description of second order d-divisibility recurrence sequences over the ring of integers $\mathbf{Z}_K$ of an algebraic number field K (cf. Theorem 1). In fact, we shall give a criterion for second order non-degenerate recurrence sequences over $\mathbf{Z}_K$ to be d-divisible for some integer $d > 1$. Further, we show (cf. Corollary 2) that a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$ is a divisibility sequence if and only if it is 2-divisible (Corollary 2). Finally, using Theorem 1 we give explicitly all second order recurrence sequences over $\mathbf{Z}$ (cf. Theorem 2) which are d-divisible for some $d > 1$.

2. Results

To state our results we need some notations. Let K be an algebraic number field and denote by $\mathbf{Z}_K$ its ring of integers. Let the sequence $\{u_h\}_{h=0}^{\infty}$ be defined by the initial terms u_0, u_1 and by the recursion

$$u_{n+2} = Au_{n+1} + Bu_n, \quad n \geq 0, \tag{1}$$

where $u_0, u_1, A, B \in \mathbf{Z}_K$ and $u_0^2 + u_1^2 \neq 0$, $B \neq 0$. Denote by α and β the zeros of the polynomial $x^2 - Ax - B$. Then we have (see e.g. [5])

$$u_n = a\alpha^n - b\beta^n \quad \text{for} \quad n = 0, 1, 2, \dots \tag{2}$$

or

$$u_n = (an + b)\alpha^n \quad \text{for} \quad n = 0, 1, 2, \dots , \tag{3}$$

according as $\alpha \neq \beta$ or $\alpha = \beta$. Further, we have $a = \dfrac{u_1 - u_0\beta}{\alpha - \beta}$ and $b = \dfrac{u_1 - u_0\alpha}{\alpha - \beta}$ in (2), and $a = \dfrac{u_1 - u_0\alpha}{\alpha}$ and $b = u_0$ in (3).

The sequence $\{u_h\}_{h=0}^{\infty}$ is called *degenerate* if α/β is a root of unity, and *non-degenerate* otherwise.

Theorem 1. *Let $\{u_h\}_{h=0}^{\infty}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$ with the parameters specified above. (A) If there exists an integer $d > 1$ and an n_0 such that $u_n | u_{nd}$ in $\mathbf{Z}_K$ for all $n \geq n_0$, then $b^{d-1} = a^{d-1}$. (B) Conversely, if $b^{d-1} = a^{d-1}$ for some integer $d > 1$, then $\{u_h\}$ is d-divisible.*

In other words, under the assumptions of Theorem 1 $\{u_h\}$ is d-divisible if and only if $b^{d-1} = a^{d-1}$.

If the assumptions in (A) of Theorem 1 hold for $d = 2$ then we get that $b = a$. Further, in this case $u_0 = 0$, $b = a = \dfrac{u_1}{\alpha - \beta}$ and hence $\{u_h\}$ is divisible. Conversely, if $u_0 = 0$ then $b = a$. Thus we have the following

Corollary 1. *Let $\{u_h\}_{h=0}^{\infty}$ be as in Theorem 1. (A) Assume that there exists an n_0 such that $u_n|u_{2n}$ in $\mathbf{Z}_K$ for all $n \geq n_0$. Then $u_0 = 0$ and*

$$u_n = u_1 \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for} \quad n = 0, 1, 2, \dots .$$

(B) Conversely, if $u_0 = 0$ then $\{u_h\}$ is divisible.

This implies the following

Corollary 2. *Let $\{u_h\}_{h=0}^{\infty}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$. $\{u_h\}$ is a divisibility sequence if and only if it is 2-divisible.*

The degree of $K(\alpha)$ over $\mathbf{Q}$ is at most $2k$, where k denotes the degree of K over $\mathbf{Q}$. Hence, in Theorem 1, there exist only finitely many possibilities for a/b (if $b \neq 0$) and b/a (if $a \neq 0$) which are easily computable if K is given. We shall carry out this explicitly only for $K = \mathbf{Q}$. Using Theorem 1, we shall list in Theorem 2 below all second order (degenerate and non-degenerate) recurrence sequences over $\mathbf{Z}$ which are d-divisible for some $d > 1$.

Theorem 2. *Let $\{u_h\}_{h=0}^{\infty}$ be a second order recurrence sequence over $\mathbf{Z}$ with the parameters $u_0, u_1, A, B \in \mathbf{Z}$, $u_0^2 + u_1^2 \neq 0$, $B \neq 0$ specified above, and let $d > 1$ be an integer. The sequence $\{u_h\}$ is d-divisible if and only if there exist $e, f \in \mathbf{Z}$ such that at least one of the following cases holds:*

(i) $u_0 = 0$, *d arbitrary;*
(ii) $A = 0$, *d odd;*
(iii) $A = 0$, $u_1 | u_0 B^{d/2}$, *d even;*
(iv) $u_0 A = 2u_1$, $A^2 + 4B = 0$, *d arbitrary;*
(v) $u_0 A = 2u_1$, *d odd;*
(vi) $A = 2e$, $B = -2e^2$, $e \neq 0$, $d \equiv 1 \pmod 4$;
(vii) $A = 2e$, $B = -2e^2$, $e \neq 0$, $u_r | B^{2t} u_{r_0}$ *for all integers* r, t, r_0 *with* $1 \leq r \leq 3$, $dr = 4t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 3$ *and* $d \not\equiv 1 \pmod 4$;
(viii) $A = 2e$, $B = -(e^2 + f^2)$, $f \neq 0$, $e \neq \pm f$, $u_1 = u_0(e \pm f)$ *and* $d \equiv 1 \pmod 4$;
(ix) $A = -f$, $B = -f^2$, $f \neq 0$, $d \equiv 1 \pmod 3$;
(x) $A = -f$, $B = -f^2$, $f \neq 0$, $u_r | f^{3t} u_{r_0}$ *for all integers* r, t, r_0 *with* $1 \leq r \leq 2$, $dr = 3t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 2$ *and* $d \not\equiv 1 \pmod 3$;
(xi) $A = 2e - f$, $B = -(e^2 - ef + f^2)$, $e \neq 0$, $f \neq 0$, $e \neq \pm f$, $u_0 \neq 0$, $u_1/u_0 \in \{e - f, e\}$, $d \equiv 1 \pmod 3$;
(xii) $A = 3f$, $B = -3f^2$, $f \neq 0$, $d \equiv 1 \pmod 6$;
(xiii) $A = 3f$, $B = -3f^2$, $f \neq 0$, $u_r | B^{3t} u_{r_0}$ *for all integers* r, t, r_0 *with*

$$\begin{array}{ll} & 1 \le r \le 5,\ dr = 6t + r_0,\ t \ge 0,\ 0 \le r_0 \le 5 \text{ and } d \not\equiv 1 \ (\textit{mod } 6); \\ \text{(xiv)} & A = 2e - f,\ B = -(e^2 - ef + f^2),\ e \ne \pm f,\ 2f,\ u_0 \ne 0, \\ & u_1/u_0 \in \{e + f, e - 2f\} \text{ and } d \equiv 1 \ (\textit{mod } 6). \end{array}$$

It follows from Theorem 2 (see also its proof) that in cases (ii) – (iv), (vi), (vii), (ix), (x), (xii) and (xiii), the sequence $\{u_h\}$ is degenerate. In case (i), it is easy to give an example both for the degenerate and for the non-degenerate case. In the other cases $\{u_h\}$ is non-degenerate. More precisely, it is easy to deduce from Theorems 1, 2 and Corollary 2 that there exist only the following five types of second order non-degenerate d-divisibility sequences $\{u_h\}$ over $\mathbf{Z}$;

$\{u_h\}$ is 2-divisible; then it is divisible (case (i));

$\{u_h\}$ is 2-non-divisible but 3-divisible; then it is $(2k + 1)$-divisible for every $k \in \mathbf{N}$ (case (v));

$\{u_h\}$ is 2-non-divisible but 4-divisible; then it is $(3k + 1)$-divisible for every $k \in \mathbf{N}$ (case (xi));

$\{u_h\}$ is 3-non-divisible but 5 divisible; then it is $(4k + 1)$-divisible for every $k \in \mathbf{N}$ (case (viii));

$\{u_h\}$ is 3 and 4 -non-divisible but 7 -divisible; then it is $(6k + 1)$-divisible for every $k \in \mathbf{N}$ (case (xiv)).

3. Proofs

PROOF *of Theorem 1.* First we prove assertion (A). Let $\{u_h\}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$, satisfying the assumptions made in (A) of Theorem 1. Using the notations of Section 2 we have $\alpha \ne \beta$, hence u_n satisfies (2).

An easy computation shows that

$$(4)\quad (a\alpha^n - b\beta^n)\frac{\alpha^{dn} - \beta^{dn}}{\alpha^n - \beta^n} = a\alpha^{dn} - b\beta^{dn} + (a - b)(\alpha\beta)^n \frac{\alpha^{(d-1)n} - \beta^{(d-1)n}}{\alpha^n - \beta^n}$$

for all integers $n, d \ge 1$. Put $L = K(\alpha)$, and denote by $\mathbf{Z}_L$ the ring of integers of L. It is clear that

$$\frac{\alpha^{jn} - \beta^{jn}}{\alpha^n - \beta^n} \in \mathbf{Z}_K \quad \text{for every integer} \quad j \ge 1.$$

Hence, if $u_n | u_{dn}$ then, by (4),

$$(a - b)(\alpha\beta)^n \frac{\alpha^{(d-1)n} - \beta^{(d-1)n}}{\alpha^n - \beta^n} \in \mathbf{Z}_K$$

and

$$u_n \mid (a-b)(\alpha\beta)^n(\alpha^{(d-1)n} - \beta^{(d-1)n}) \quad \text{in } \mathbf{Z}_L. \tag{5}$$

Further, we note that $a - b = u_0$, whence $a - b \in \mathbf{Z}_K$.

For a non-zero ideal $\mathcal{A}$ of $\mathbf{Z}_L$, let $P(\mathcal{A})$ denote the maximum of the rational primes lying below the prime ideal divisors of $\mathcal{A}$. Further, we put $P(0) = P(1) = 1$. By a result of MAHLER [3], $P(u_n)$ is not bounded as $n \to \infty$. Hence there exists a number n_1 such that $P(u_n) > \max\{P(a-b), P(\alpha\beta)\}$ for infinitely many $n \geq n_1$. Let $n \geq \max\{n_0, n_1\}$ with this property, and let $\wp_n$ be a prime ideal divisor of u_n with $P(\wp_n) = P(u_n)$. Then it follows from (5) that

$$\wp_n \mid \alpha^{(d-1)n} - \beta^{(d-1)n} \quad \text{in } \mathbf{Z}_L. \tag{6}$$

Put

$$A_i = (\alpha-\beta)^{i-1}(b^{i-1}\alpha^{(d-i)n} - a^{i-1}\beta^{(d-i)n}) \quad \text{for } i = 1, \dots, d.$$

Then $A_i \in \mathbf{Z}_L$ for each i. Further, $b(\alpha - \beta) \in \mathbf{Z}_L$. It is easy to see that

$$(\alpha-\beta)^{i-1}\beta^{(d-i)n}a^{i-2}(a\alpha^n - b\beta^n) - b(\alpha-\beta)A_{i-1} = -\alpha^n A_i \tag{7}$$

for each integer i with $2 \leq i \leq d$. We prove now that

$$\wp_n | A_i \quad \text{in} \quad \mathbf{Z}_L \tag{8}$$

for each integer i with $1 \leq i \leq d$. By (6), (8) is true for $i = 1$. Assume that it is true for $i - 1 \geq 0$. Then by the definition of $\wp_n$ and by the induction hypothesis, $\wp_n$ divides the element on the left-hand side of (7). Since $\wp_n \nmid \alpha^n$ in $\mathbf{Z}_L$, it must divide A_i in $\mathbf{Z}_L$ on the right-hand side of (7), and (8) is proved.

Setting $i = d$ in (8) we get

$$\wp_n \mid (\alpha-\beta)^{d-1}(b^{d-1} - a^{d-1}) \quad \text{in} \quad \mathbf{Z}_L.$$

But $P(\wp_n)$ can be arbitrarily large, hence $b^{d-1} = a^{d-1}$ which proves the assertion in (A).

Conversely, suppose now that $b^{d-1} = a^{d-1}$ for some integer $d > 1$. Then $b = \zeta_{d-1}a$ with some $(d-1)$-th root of unity ζ_{d-1} from L. Then, by (2), we have

$$(9)\left\{\begin{array}{l} u_{dn} = a(\alpha^{dn} - \zeta_{d-1}\beta^{dn}) = a((\alpha^n)^d - (\zeta_{d-1}\beta^n)^d) = u_n v_n \quad \text{for all } n \geq 0, \\ \text{where} \\ v_n = \left\{\begin{array}{l} (\alpha^n)^{d-1} + (\alpha^n)^{d-2}(\zeta_{d-1}\beta^n) + \ldots + (\zeta_{d-1}\beta^n)^{d-1} \quad \text{for } n \geq 1, \\ 1 \quad \text{for } n = 0. \end{array}\right. \end{array}\right.$$

In (9), v_n is an algebraic integer. On the other hand, if $u_n \neq 0$ then, by (9), v_n belongs to K, hence $v_n \in \mathbf{Z}_K$. Thus $\{u_h\}$ is d-divisible which proves the assertion of (B).

PROOF *of Theorem 2.* Let now $\{u_h\}_{h=0}^{\infty}$ be a second order recurrence sequence over $\mathbf{Z}$ with initial terms $u_0, u_1 \in \mathbf{Z}$, $u_0^2 + u_1^2 \neq 0$, which satisfies (1) with some $A, B \in \mathbf{Z}$, $B \neq 0$, and suppose that it is d-divisible for some integer $d > 1$. Define α, β, a, b as in Section 2.

Assume first that $\{u_h\}$ is degenerate. Then $\alpha = \zeta\beta$, where ζ is a root of unity belonging to $K = \mathbf{Q}(\alpha)$. The degree of K over $\mathbf{Q}$ is at most two, hence $\zeta \in \mathcal{E} = \{\pm 1, \pm i, \pm\rho, \pm\rho^2\}$ where $\rho = \frac{-1+\sqrt{-3}}{2}$ (see e.g. [2]). We shall distinguish several cases.

Case 1. If $\alpha = \beta$, then $\alpha \in \mathbf{Z}$ and u_n may be written in the form (3). Further, $A^2 + 4B = 0$ and $\alpha = A/2$. If $a = 0$ then $u_0 A = 2u_1$ which corresponds to case (iv). This sequence is indeed d-divisible for every $d > 1$. If $b = 0$ then $u_0 = 0$ which corresponds to case (i). Then the sequence is d-divisible. Finally, suppose that a and b are different from zero. Then $\alpha(and + b)$ and $\alpha(an + b)$ are rational integers and $u_n \mid u_{dn}$ implies that $\alpha(an + b) \mid \alpha(and + b)$. From this it follows that $\alpha(an + b) \mid \alpha b(d-1)$ which is impossible if n is large enough.

Case 2. Let now $\alpha = -\beta$. Then $A = 0$ and $B = \beta^2$. We have $u_{2n} = u_0\beta^{2n}$ and $u_{2n+1} = u_1\beta^{2n}$ for $n = 0, 1, \ldots$. If d is odd then $dn \equiv n \pmod 2$, hence, indeed, $u_n | u_{dn}$ and (ii) follows. While if d is even and n is odd then $u_1\beta^{n-1} \mid u_0\beta^{dn}$ must hold. Then $\{u_h\}$ is indeed d-divisible and $\{u_h\}$ is described in case (iii).

Case 3. Let $\alpha = \pm i\beta$. Then $\alpha + \beta = \beta(1 \pm i) = A$ implies that $\beta = \frac{A}{2}(1 \mp i)$. Since β is an algebraic integer, we have $A = 2e$ and $B = -\alpha\beta = -2e^2$. Let n be an arbitrary non-negative integer, and put $n = 4v + r$ with non-negative integers v, r such that $0 \leq r < 4$. Then, by (2),

$$\begin{aligned} u_n &= a\alpha^n - b\beta^n = \beta^n(a(\pm i)^n - b) = \\ &= e^n(1 \mp i)^n(a(\pm i)^n - b) = (-1)^v B^{2v} u_r. \end{aligned} \tag{10}$$

If $d \equiv 1 \pmod 4$ then $dn \equiv n \equiv r \pmod 4$ and, by (10), $u_n | u_{dn}$, i.e. $\{u_h\}$ is indeed d-divisible, and this is case (vi). If $d \not\equiv 1 \pmod 4$ then we put $dr = 4t + r_0$ with non-negative integers t, r_0 such that $0 \leq r_0 < 4$. Then $dn = 4vd + 4t + r_0$. By (10), in this case $\{u_h\}$ is d-divisible if and only if

$$u_r | B^{2(v(d-1)+t)} u_{r_0} \qquad \begin{array}{l} \text{for all } v \geq 0 \text{ and each } r, t, r_0 \text{ such that} \\ 0 \leq r,\, r_0 < 4 \text{ and } dr = 4t + r_0. \end{array} \tag{11}$$

But for fixed r, r_0 and t, (11) holds for all $v \geq 0$ if and only if it holds for $v = 0$. Finally, (11) trivially holds for $r = 0$, hence we get case (vii).

Case 4. Next let $\alpha/\beta = \rho^j$ where $j = 1$ or 2. Then α and β belong to the ring of integers of the Eulerian number field $\mathbf{Q}(\rho)$. In this field, $\{1, \rho\}$ is an integral basis, hence we can write $\alpha = e+(-1)^j f\rho$ and $\beta = e+(-1)^j f\rho^2$ with some $e, f \in \mathbf{Z}$. Then, by $\alpha = \beta\rho^j$, we get $A = -f$, $B = -f^2$ and $\beta = \rho^j f$, $j = 1, 2$. Put $n = 3v + r$ with integers v, r such that $v \geq 0$, $0 \leq r < 3$. Then, by (2) and $(\rho^j)^3 = 1$, we have

$$u_n = \beta^n(a\rho^{jn} - b) = f^{3v} \cdot f^r \cdot \rho^{jr}(a\rho^{jr} - b) = f^{3v}u_r.$$

From now on we can proceed in a similar way as in the cases corresponding to $\alpha/\beta = \pm i$, and we get (ix) and (x) in the theorem.

Case 5. Let now $\alpha/\beta = -\rho^j$ where $j = 1$ or 2. Then we get in the same way as in the preceding case that $A = 3f$, $B = -3f^2$ and $\beta = \sqrt{-3} \cdot f(-\rho)^j$ with some rational integer $f \neq 0$. Let $n = 6v + r$ with rational integers v, r such that $v \geq 0$, $0 \leq r < 6$. Then $(-\rho^j)^6 = 1$ and (2) imply again that

$$u_n = \beta^n(a(-\rho^j)^n - b) = (\sqrt{-3}f)^{6v}u_r = B^{3v}u_r.$$

Now we can proceed as in the case above $\alpha/\beta = \pm i$, and we get cases (xii), (xiii) in our theorem.

In the sequel we suppose that $\{u_h\}_{h=0}^{\infty}$ is non-degenerate. Since $\{u_h\}$ is a second order recurrence sequence, we have $ab \neq 0$. By Theorem 1, $\{u_h\}$ is d-divisible if and only if $b = \zeta a$ with some $(d-1)$-th root of unity ζ. Then $\zeta \in \mathbf{Q}(\alpha)$, hence $\zeta \in \mathcal{E}$. Further, $b = \zeta a$ is equivalent to

$$(u_1 - u_0\beta)\zeta = u_1 - u_0\alpha. \tag{12}$$

It suffices to determine all second order non-degenerate recurrence sequences $\{u_h\}$ in $\mathbf{Z}$ having the property (12). We shall distinguish again several cases.

Case 6. Let first $\zeta = 1$. Then (12) implies $u_0 = 0$ and $\{u_h\}$ is d-divisible for every $d > 1$. This corresponds to (i) in the theorem.

Case 7. $\zeta = -1$. This appears only if d is odd. Then we get from (12) that

$$2u_1 = u_0(\alpha + \beta) = u_0 A$$

because α and β are the zeros of $x^2 - Ax - B$. This is the case (v) in the theorem.

Case 8. $\zeta = \pm i$. This is possible only if $d \equiv 1 \pmod 4$.

Then $K = \mathbf{Q}(\alpha) = \mathbf{Q}(i)$ is the Gaussian number field. Hence $\alpha = e + if$ and $\beta = e - if$ with suitable $e, f \in \mathbf{Z}\setminus\{0\}$ for which $e \neq \pm f$.

For $\zeta = i$ and $\zeta = -i$, (12) implies $u_1 = u_0(e-f)$ and $u_1 = u_0(e+f)$, respectively. Furthermore, we have in both cases $A = 2e$ and $B = -(e^2+f^2)$ which corresponds to case (viii) of the theorem.

Case 9. $\zeta = \rho^j$ where $j = 1$ or 2. Then $d \equiv 1 \pmod 3$ and $K = \mathbf{Q}(\alpha) = \mathbf{Q}(\rho)$ is the Eulerian number field. Thus $\alpha = e + f\rho$ and $\beta = e + f\rho^2$ with suitable integers $e, f \in \mathbf{Z}\setminus\{0\}$ for which $e \neq \pm f, 2f$. Hence $A = 2e - f$ and $B = -(e^2 - ef + f^2)$. Using again (12), we get

$$u_1 = \begin{cases} u_0(e-f) & \text{if } \zeta = \rho \\ u_0 e & \text{if } \zeta = \rho^2, \end{cases}$$

which corresponds to (xi) in the theorem.

Case 10. $\zeta = -\rho^j$ where $j = 1$ or 2. Then $d \equiv 1 \pmod 6$ and $K = \mathbf{Q}(\alpha) = \mathbf{Q}(\rho)$. Taking again $\alpha = e + f\rho$ and $\beta = e + f\rho^2$ with some $e, f \in \mathbf{Z}\setminus\{0\}$, we get for e, f the restrictions $e \neq \pm f, 2f$. A and B have the same form as in case 9. Finally, using (12) we get

$$u_1 = \begin{cases} u_0(e+f) & \text{if } \zeta = -\rho \\ u_0(e-2f) & \text{if } \zeta = -\rho^2, \end{cases}$$

which corresponds to case (xiv) in the theorem. This completes the proof of Theorem 2.

Acknowledgements. We are indebted to the referee for calling our attention to some minor inaccuracies in the manuscript.

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K. GYŐRY and A. PETHŐ
MATHEMATICAL INSTITUTE
LAJOS KOSSUTH UNIVERSITY
H-4010 DEBRECEN, P.O. BOX 12
HUNGARY

(Received December 7, 1990)