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On the difference of integer-valued additive functions

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1. Introduction

A classical theorem of ERDŐS [4] asserts that if a real-valued additive function satisfies $f(n+1) - f(n) \to 0$, then it must be of the form $f(n) = c \log n$. Many generalizations and analogs have been found since then. An important one is the following (still unpublished) result of E. WIRSING: if a multiplicative function satisfies $|g(n)| = 1$, $g(n+1) - g(n) \to 0$, then it must be of the form $g(n) = n^{ic}$ with some real constant c.

DARÓCZY and KÁTAI [3] found the following common generalization of these results: if G is a locally compact, compactly generated abelian group (written additively) and a G–valued additive arithmetical function f satisfies $f(n+1) - f(n) \to 0$, then f is the restriction to the set of integers of a continuous homomorphism from the multiplicative group of positive real numbers to G.

DARÓCZY and KÁTAI's proof is based on an application of WIRSING's theorem to the functions $g(n) = \chi(f(n))$, where χ runs over the (continuous) characters of G. This approach cannot handle groups that are not separated by their characters. AJTAI, HAVAS and KOMLÓS [1] have shown that there are commutative groups with a Hausdorff topology that do not admit any nontrivial continuous character. This leaves the question of existence of nontrivial group-valued additive functions with $f(n+1) - f(n) \to 0$ open. We show that such functions do indeed exist, even for the simplest G, the additive group of integers.

Theorem 1. *There is an integer-valued completely additive function f, not identically 0, and a Hausdorff topology T on the set of integers which makes it a topological group with the operation of addition, such that*

$$f(n+1) - f(n) \to 0 \text{ in } T\,.$$

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Before starting the proof we show why our function does not fall within the frames of a suitable extension of DARÓCZY and KÁTAI's theorem.

Let G be a topological group. It seems reasonable to call a G–valued additive function *regular*, if it can be extended to a continuous homomorphism from the multiplicative group of rational numbers to G. (The existence of an extension to the group of positive reals depends on completeness properties of G.) We show that our function does not possess such an extension.

1.1. Statement. *Let G be the group of integers with an arbitrary Hausdorff group topology. There is no nontrivial continuous homomorphism from $\mathbf{Q}^*$, the multiplicative group of positive rational numbers with the usual topology, to G.*

PROOF. Suppose that there is such a homomorphism φ. Take a prime p such that $\varphi(p) \neq 0$, and two other primes q and r. Write $\varphi(p) = a$, $\varphi(q) = b$, $\varphi(r) = c$; we know $a \neq 0$. Define $m = p^b q^{-a}$, $n = p^c r^{-a}$. Here m contains q but not r while n contains r but not q (they may and may not contain p), thus they are different from 1 and not powers of a common base, therefore $(\log m)/(\log n)$ is irrational. This implies that the numbers $m^u n^v$, u, v integers, are dense in $\mathbf{Q}$. Since φ vanishes on this dense set, it must be identically zero. □

An even stronger requirement would be that the function cannot be extended to an algebraical homomorphism, in other words, that it is not completely additive. This is, however, impossible.

1.2. Statement. *An additive function with values in an arbitrary commutative topological group that satisfies $f(n+1) - f(n) \to 0$ must be completely additive.*

PROOF. To prove complete additivity it is sufficient to show $f(p^{k+1}) = f(p^k) + f(p)$ for every prime p and positive integer k. Now observe that the equality

$$f(p^{k+1}) - f(p^k) - f(p) = \big(f(p^k n+1) - f(p^k n)\big) - \big(f(p^{k+1} n+p) - f(p^{k+1} n)\big)$$

holds for every n coprime to p, and the right side tends to 0 as $n \to \infty$. □

2. Proof of the Theorem

2.1. Definition. A Hausdorff topology on the set of integers is called an *arithmetical topology*, if it turns the set of integers with the operation of addition into a topological group.

2.2. Definition. We call a sequence a_n of integers *nullpotent*, if there is an arithmetical topology in which $a_n \to 0$.

With this terminology, we need to construct a nonzero completely additive function for which the sequence $f(n+1)-f(n)$ is nullpotent. We prove slightly more.

Theorem 2. *There is a positive-valued function $F(n)$ such that the sequence $f(n+1)-f(n)$ is nullpotent for every completely additive function f which satisfies*

$$|f(q)| > F(q) \max_{p<q} |f(p)| \tag{2.1}$$

for all but finitely many primes q.

We quote the following arithmetical description of nullpotency from RUZSA [5].

2.3. Lemma. *A sequence a_n is nullpotent if and only if for every integer $u \neq 0$ and positive integer k the equation*

$$u = e_1 a_{n_1} + e_2 a_{n_2} + \cdots + e_k a_{n_k}, \quad e_n = \pm 1 \tag{2.2}$$

has only finitely many primitive solutions. Here a solution of (2.2) is called primitive, if none of the $2^k - 1$ nonempty subsums is 0.

Substituting $a_n = f(n+1) - f(n)$ into (2.2), we obtain the equation

$$u = \sum e_i (f(n_i + 1) - f(n_i)). \tag{2.3}$$

If we extend f to rational numbers naturally by putting $f(a/b) = f(a) - f(b)$, then (2.3) can be rewritten as

$$u = f(Q), \quad Q = \prod \left(\frac{n_i + 1}{n_i} \right)^{e_i} = \prod \left(\frac{m_i + e_i}{m_i} \right), \tag{2.4}$$

where

$$m_i = \begin{cases} n_i & \text{if } e_i = 1, \\ n_i + 1 & \text{if } e_i = -1. \end{cases} \tag{2.5}$$

For the proof the following lemma on prime factors of such products Q is fundamental.

2.4. Lemma. *Let $m_1 \leq m_2 \leq \cdots \leq m_k$ be positive integers, $b_1, \ldots, b_k$ integers such that $|b_i| \leq A$ and $m_i + b_i > 0$ $(i = 1, \ldots, k)$. Write*

$$Q = \prod_{i=1}^{k} \left(\frac{m_i + b_i}{m_i} \right).$$

Assume that all prime factors in the numerator and denominator of Q are $\leq P$ and that

$$\prod_{i=j}^{k} \left(\frac{m_i + b_i}{m_i} \right) \neq 1 \quad (j = 1, \ldots, k). \tag{2.6}$$

Then we have $m_k \leq G(k, P, A)$, an effectively computable number depending only on k, P and A.

This lemma, which will be proved in the next section, finishes our preparation to the proof of the theorem.

PROOF OF THEOREM 2. We put $F(q) = 4qG(q, q, 1)$ with the function G of Lemma 2.4. We have to prove that (2.3) has only finitely many primitive solutions. Let q_0 be such a number that (2.1) holds for $q \geq q_0$, and also that $f(p)$ is not identically 0 for $p < q_0$, in which case (2.1) also means $|f(q)| > F(q)$ $(q > q_0)$.

Without restricting the generality we may assume that the m_i given by (2.5) are increasing. Consider a primitive solution and let q be the largest prime that occurs in the numerator or denominator of the number Q given in (2.4). From Lemma 2.4 we infer $m_k \leq G(k, q, 1)$; condition (2.6) follows from the primitivity of (2.3) (in fact, we needed only the k interval-subsums).

Let $K = \max(k, |u|, q_0)$. Assume first $q > K$. We have obviously

$$|f(Q)| \geq |f(q)| - r \max_{p<q} |f(p)| \geq |f(q)|(1 - r/F(q)), \tag{2.7}$$

where r is the total number of primes, counted with multiplicity, that occur in any of the numbers n_i or n_i+1. Since $k < q$, we have $n_i \leq G(q, q, 1)$, thus the number of prime factors in any of these numbers is at most $G(q, q, 1)$ and we have $r \leq 2kG(q, q, 1) < F(q)/2$. Hence (2.7) yields

$$|u| = |f(Q)| \geq |f(q)|/2 \geq F(q)/2 > q > |u|,$$

a contradiction.

Thus we have $q \leq K$, and from Lemma 2.4 we infer $m_k \leq G(k, K, 1)$, a finite number of choices. □

3. Proof of Lemma 2.4

We use the following result of BAKER [2] in the form given by SHOREY et al. [6], p. 66 (we specialized the formulation to integers).

3.1. Lemma. *Let $\alpha_1, \dots, \alpha_\ell$ be positive integers, $\ell \geq 2$, $\alpha_i \leq A_i \geq 4$ for $i = 1, \dots, \ell$. Put*

$$\Omega = \prod_{i=1}^{\ell} \log A_i, \quad \Omega' = \prod_{i=1}^{\ell-1} \log A_i .$$

Let $b_1, \dots, b_\ell$ be integers with $|b_j| \leq B \geq 4$. Then either $\alpha_1^{b_1} \dots \alpha_\ell^{b_\ell} = 1$ or

$$\log |\alpha_1^{b_1} \dots \alpha_\ell^{b_\ell} - 1| > -\ell^{c\ell} \Omega \log \Omega' \log B \tag{3.1}$$

with an absolute constant c.

PROOF OF LEMMA 2.4. We use Vinogradov's symbol $\ll$ where the implicit constant may depend on k, A and P.

Write Q in the form $Q = p_1^{t_1} \dots p_s^{t_s}$ with distinct primes $p_1, \dots, p_s$ and integers t_i. We have

$$|t_i| \leq 2k \log(m_k + A) \ll \log m_k$$

for all i. The lemma above yields

$$\log |Q - 1| \gg - \log \log m_k,$$

while a direct computation gives $|Q - 1| \ll 1/m_1$. Combining the two we obtain

$$\log m_1 \ll \log \log m_k.$$

Now we prove the inequalities

$$\log m_j \ll (\log \log m_k)^j \tag{3.2}$$

by induction on j. Suppose it holds for $1, \dots, j-1$. On one hand we have

$$\left| \frac{m_j + b_j}{m_j} \frac{m_{j+1} + b_{j+1}}{m_{j+1}} \dots \frac{m_k + b_k}{m_k} - 1 \right| \ll \frac{1}{m_j} . \tag{3.3}$$

On the other hand

$$\left| \frac{m_j + b_j}{m_j} \frac{m_{j+1} + b_{j+1}}{m_{j+1}} \dots \frac{m_k + b_k}{m_k} - 1 \right| = \left| Q \prod_{i=1}^{j-1} \frac{m_i}{m_i + b_i} - 1 \right| .$$

To this expression we apply our Lemma with $\ell = s + 1$, $\alpha_\ell = \prod_{i=1}^{j-1} (m_i/(m_i + b_i))$, $b_\ell = 1$. By the induction hypothesis we have

$$\log A_\ell \leq j \log(m_{j-1} + |b_{j-1}|) \ll (\log \log m_k)^{j-1} ,$$

hence

$$\log \left| \frac{m_j + b_j}{m_j} \frac{m_{j+1} + b_{j+1}}{m_{j+1}} \dots \frac{m_k + b_k}{m_k} - 1 \right| \gg$$
$$\gg -(\log\log m_k)^{j-1} \log\log m_k .$$

On combining this inequality with (3.3) we obtain (3.2) for j.

Finally, the case $j = k$ of (3.2) means

$$\log m_k \ll (\log\log m_k)^k ,$$

which implies $m_k \ll 1$. □

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