

On ideals and extensions of near-rings

By STEFAN VELDSMAN (Port Elizabeth)

Abstract. Given a chain of ideals $J \triangleleft I \triangleleft N$ in a near-ring N , we consider necessary and sufficient conditions on J , I , N , N/I and I/J respectively to ensure that $J \triangleleft N$.

§1. Introduction

Near-rings considered will be right near-rings; the variety of all near-rings will be denoted by $\mathcal{V}$ and the subvariety of 0-symmetric near-rings will be denoted by $\mathcal{V}_0$. To facilitate discussions, the conditions mentioned in the Abstract will be formulated in terms of a subclass $\mathcal{M}$ of $\mathcal{V}$.

The class $\mathcal{M}$ satisfies condition

- (F) If $J \triangleleft I \triangleleft N$ and $I/J \in \mathcal{M}$, then $J \triangleleft N$
- (G) If $J \triangleleft I \triangleleft N$ and $J \in \mathcal{M}$, then $J \triangleleft N$
- (H) If $J \triangleleft I \triangleleft N$ and $I \in \mathcal{M}$, then $J \triangleleft N$
- (K) If $J \triangleleft I \triangleleft N$ and $N \in \mathcal{M}$, then $J \triangleleft N$
- (L) If $J \triangleleft I \triangleleft N$ and $N/I \in \mathcal{M}$, then $J \triangleleft N$.

It is our purpose here to describe the near-rings in $\mathcal{M}$ for each of the above five cases. For rings, this has been done, see [8] or SANDS [3]. For the purpose of comparison, we recall:

In the variety of rings, a subclass $\mathcal{M}$ satisfies condition:

- (F) if and only if the rings in $\mathcal{M}$ are quasi semiprime, i.e. if $R \in \mathcal{M}$ and $xR = 0$ or $Rx = 0$ ($x \in R$), then $x = 0$ (or equivalently, R has zero middle annihilator, i.e. $RxR = 0$ ($x \in R$) implies $x = 0$).
- (G) if and only if $R^2 = R$ for all $R \in \mathcal{M}$.
- (H) if and only if every ideal I of $R \in \mathcal{M}$ is invariant under all double homothetisms of R (cf RÉDEI [2]).
- (K) if and only if for all $R \in \mathcal{M}$ and for all $a \in R$, $(a) = (a)^2 + \mathbf{Z}a$ where (a) is the ideal in R generated by a and $\mathbf{Z}$ is the integers.

(L) if and only if $\mathcal{M} = \{0\}$.

If N is a near-ring, then N^+ will denote the underlying group. For nearrings N_i , $i = 1, 2, \dots, k$ and subsets $U_i \subseteq N_i$, $(U_1, U_2, \dots, U_k)$ denotes the subset $\{(u_1, u_2, \dots, u_k) \mid u_i \in U_i\}$ of $N_1 \times N_2 \times \dots \times N_k$.

§2. On condition (F)

In the variety of 0-symmetric near-rings, this problem has been settled in [9]: A class of near-rings $\mathcal{M}$ in $\mathcal{V}_0$ satisfies condition (F) if and only if every near-ring in $\mathcal{M}$ is quasi semi-equiprime. A near-ring N is *quasi semi-equiprime* if $xN = 0$ ($x \in N$) implies $x = 0$ and whenever $\theta : I \rightarrow N$ is a surjective homomorphism with $I \triangleleft A$ then $x - y \in \ker\theta$ ($x, y \in I$), implies $ax - ay \in \ker\theta$ for all $a \in A$.

In the variety of all near-rings, a complete description is still outstanding. The construction in [7] shows that $\mathcal{M}$ does not contain any non zero constant near-rings. In fact, it is conjectured that $\mathcal{M} = \{0\}$; the strongest motivation for this coming from the example in [6] which shows that any class which contains the two element field cannot satisfy condition (F).

§3. On condition (G)

This case can quickly be disposed of using a construction given in [5] which resembles one given by BETSCH and KAARLI [1]. Let N be a near-ring and let K be the near-ring with $K^+ = N^+ \oplus N^+ \oplus N^+$ and with multiplication

$$(a, b, c)(x, y, z) = \begin{cases} (b, 0, cz) & \text{if } y \text{ and } z \text{ are non-zero} \\ (0, 0, cz) & \text{otherwise.} \end{cases}$$

Apart from verifying the associativity of the multiplication as well as the right distributivity over the addition, it is straightforward to see that $N \cong (0, 0, N) \triangleleft (N, 0, N) \triangleleft K$, $(0, 0, N) \triangleleft K$ if and only if $N = 0$ and K is 0-symmetric if and only if N is 0-symmetric.

Theorem 3.1. *In either one of $\mathcal{V}$ or $\mathcal{V}_0$, a subclass $\mathcal{M}$ satisfies condition (G) if and only if $\mathcal{M} = \{0\}$.*

PROOF. Suppose $\mathcal{V}$ satisfies condition G and let $N \in \mathcal{V}$. Then $N \cong (0, 0, N) \triangleleft (N, 0, N) \triangleleft K$; hence $(0, 0, N) \triangleleft K$, which yields $N = 0$.

§4. On condition (L)

As in the variety of rings, a subclass $\mathcal{M}$ of $\mathcal{V}$ or $\mathcal{V}_0$ satisfies condition (L) if and only if $\mathcal{M} = \{0\}$. To verify this for near-rings, we need two constructions:

4.1 Let N be a near-ring and let G be the group $G = N^+ \oplus U$ where U is any non-zero group. As is well-known, N can be identified with a subnear-ring of $M(G) = \{f \mid f : G \rightarrow G \text{ a function}\}$ via $\theta : N \rightarrow M(G)$,

$$\theta(n) = \theta_n : G \rightarrow G, \quad \theta_n(g) = \begin{cases} ng & \text{if } g \in N \\ n & \text{if } g \in G \setminus N. \end{cases}$$

Let K_1 be the near-ring with $K_1^+ = N^+ \oplus M(G)^+$ and multiplication

$$(n, g)(m, h) = (nm, nh).$$

Let $X = \{f \in M(G) \mid f(U) \subseteq U\}$. Then $(0, X) \triangleleft (0, M(G)) \triangleleft K_1$ and $K_1/(0, M(G)) \cong N$. Note that $(0, X)$ is a right ideal in K_1 . It is a left ideal of K_1 if and only if N is constant. Indeed, $(0, X)$ is a left ideal of K_1 if and only if $n(g+h) - ng \in X$ for all $n \in N$, $g \in M(G)$ and $h \in X$. Let g and h be the functions defined by

$$g(x) = \begin{cases} x & \text{if } x \in N \\ 0 & \text{if } x \in G \setminus N \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 0 & \text{if } x \in N \\ x & \text{if } x \in G \setminus N. \end{cases}$$

Clearly $h \in X$; thus if $(0, X) \triangleleft K_1$, then $n(g+h) - ng \in X$. Thus, for $0 \neq u \in U$, $n(g(u) + h(u)) - ng(u) \in U$,

$$\text{i.e. } n(u) - n(0) \in U$$

$$\text{i.e. } n - n0 \in U \cap N = \{0\}; \text{ hence } n = n0.$$

Conversely, if N is constant, then $(0, X) \triangleleft K_1$. If N is 0-symmetric, and we replace $M(G)$ above with $M_0(G)$, then everything above stays valid except in this case $(0, X) \triangleleft K_1$ if and only if $N = 0$.

4.2 Let N be a near-ring and let G be any group which properly contains N^+ . We regard N as a subnear-ring of $M(G)$. Let K_2 be the near-ring with $K_2^+ = N^+ \oplus M(G)^+$ and multiplication $(n, f)(m, g) = (nm, fg)$. Then $(0, M_0(G)) \triangleleft (0, M(G)) \triangleleft K_2$ and $K_2/(0, M(G)) \cong N$. Furthermore, $(0, M_0(G))$ is a left ideal in K_2 , but it is an ideal if and only if N is 0-symmetric. Indeed, if it is a right ideal, then $(0, 1_G)(n, 0) \in (0, M_0(G))$ which gives $n0 = 0$. The converse is obvious.

Theorem 4.3. *In either one of $\mathcal{V}$ or $\mathcal{V}_0$, a subclass $\mathcal{M}$ satisfies condition (L) if and only if $\mathcal{M} = \{0\}$.*

PROOF. Firstly, if $\mathcal{M}$ is a subclass in $\mathcal{V}_0$ which satisfies condition (L) and $N \in \mathcal{M}$, we have from the construction in 4.1 that $N = 0$. If $\mathcal{M}$ is a

subclass in $\mathcal{V}$ which satisfies condition (L) and $N \in \mathcal{M}$, the construction in 4.1 shows that N is constant and the construction in 4.2 shows that N is 0-symmetric. Hence $N = 0$.

§5. On condition (K)

Principal ideals in a near-ring, contrary to the ring case, have no nice finite description. This seems to be the most serious obstacle in describing the near-rings in a class $\mathcal{M}$ which satisfies condition (K). For a near-ring N and $a \in S \subseteq N$, S a subnear-ring of N , the ideal in S generated by a will be denoted by $\langle a, S \rangle$.

Theorem 5.1. *Let $\mathcal{M}$ be a class of near-rings. Then $\mathcal{M}$ satisfies condition (K) if and only if $\langle a, N \rangle = \langle a, \langle a, N \rangle \rangle$ for all $a \in N$, $N \in \mathcal{V}$.*

PROOF. Firstly, if $a \in N \in \mathcal{M}$ and $\mathcal{M}$ satisfies condition (K), then $\langle a, \langle a, N \rangle \rangle \triangleleft N$. Hence $\langle a, N \rangle = \langle a, \langle a, N \rangle \rangle$. Conversely, if the condition is satisfied, choose $J \triangleleft I \triangleleft N \in \mathcal{M}$ and let $j \in J$, $n, m \in N$. Then $\langle j, N \rangle \cap J \subseteq \langle j, N \rangle \subseteq I$; hence

$$\langle j, N \rangle = \langle j, \langle j, N \rangle \rangle \subseteq \langle j, N \rangle \cap J \subseteq J.$$

Thus $n + j - n$, jn and $n(m + j) - nm \in J$ which yields $J \triangleleft N$.

§6. On condition (H)

We start with finding the near-ring analogue of the Schreier group extensions or the Everett ring extensions (cf RÉDEI [2]): Given near-rings A and B determine all near-rings N such that N is an extension of A by B , i.e. $A \triangleleft N$ and $N/A = B$. This problem has earlier been settled for composition rings and 0-symmetric near-rings by STEINEGGER [4]. Strictly speaking, it involves finding all triples (ζ, N, η) where

$$0 \longrightarrow A \xrightarrow{\zeta} N \xrightarrow{\eta} B \longrightarrow 0$$

is a short exact sequence. Two extensions (ζ, N, η) and (ζ', N', η') of A by B are *equivalent* if there exists an isomorphism $\chi : N \rightarrow N'$ such that the diagram

commutes. χ is called an *equivalence isomorphism*. In order to simplify notation and discussions, our exposition will not be as rigorous as required above; instead, we identify A with $\zeta(A)$ and $N/\zeta(A)$ with B . In such a case, it means $\chi(a) = a$ for all $a \in A$ and $\chi(n) + A = n + A$ for all $n \in N$. The elements of A and B will always be taken as $A = \{a, b, c, \dots\}$ and $B = \{\alpha, \beta, \gamma, \dots\}$ respectively with 0 denoting the additive identity of both A and B .

Consider a quintuple of functions $(F, [-, -], G, H, \langle -, - \rangle)$ with $F : B \rightarrow M(A)$, $[-, -] : B \times B \rightarrow A$, $G : A \times B \rightarrow M(A)$, $H : B \times B \rightarrow M(A)$ and $\langle -, - \rangle : B \times B \rightarrow A$ which satisfy the *initial conditions*

$$\begin{aligned} F(\alpha) &\in M_0(A), F(0) = 1_A \\ [\alpha, 0] &= 0 = [0, \alpha] \\ G(b, \beta) &\in M_0(A), G(b, 0)(a) = ab \\ H(0, \beta) &= 0; H(\alpha, \beta) \in M_0(A) \\ \langle 0, \beta \rangle &= 0. \end{aligned}$$

On the cartesian product $A \times B$ define two operations by

$$(a, \alpha) + (b, \beta) = (a + F(\alpha)(b) + [\alpha, \beta], \alpha + \beta)$$

and

$$(a, \alpha)(b, \beta) = (G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle, \alpha\beta).$$

We say these two operations are the operations induced by the function quintuple. $A \times B$ together with these two operations is called an *E-sum of A and B* w.r.t. the quintuple $(F, [-, -], G, H, \langle -, - \rangle)$ and is denoted by $A \sharp B$. The two functions $[-, -]$ and $\langle -, - \rangle$ are called *the additive and multiplicative factor systems* of the *E-sum* respectively.

Although A is (as a near-ring) isomorphic to $(A, 0)$ (via $a \rightarrow (a, 0)$), in general $A \sharp B$ may have no particular structure w.r.t. the induced operations. A function quintuple $(F, [-, -], G, H, \langle -, - \rangle)$ is called an *amicable system for A w.r.t. B* if it satisfies the following conditions for all $a, b, c \in A$ and $\alpha, \beta, \gamma \in B$:

- (E1) $F(\alpha) \in \text{End}(A^+)$
- (E2) $[\alpha, \beta] + F(\alpha + \beta)(c) = F(\alpha) (F(\beta)(c)) + [\alpha, \beta]$
- (E3) $[\alpha, \beta] + [\alpha + \beta, \gamma] = F(\alpha) ([\alpha, \beta]) + [\alpha, \beta + \gamma]$
- (E4) $G(c, \gamma) \in \text{End}(A^+)$
- (E5) $G(c, \gamma) (F(\alpha)(b)) + G(c, \gamma) ([\alpha, \beta]) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma \rangle$
 $= H(\alpha, \gamma)(c) + \langle \alpha, \gamma \rangle + F(\alpha\gamma) (G(c, \gamma)(b)) + F(\alpha\gamma) (H(\beta, \gamma)(c))$
 $+ F(\alpha\gamma) (\langle \beta, \gamma \rangle) + [\alpha\gamma, \beta\gamma]$

$$\begin{aligned}
\text{(E6)} \quad & G(c, \gamma) (H(\alpha, \beta)(b)) + G(c, \gamma) (\langle \alpha, \beta \rangle) + H(\alpha\beta, \gamma)(c) + \langle \alpha\beta, \gamma \rangle \\
& = H(\alpha, \beta\gamma) (G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle) + \langle \alpha, \beta\gamma \rangle \\
\text{(E7)} \quad & G(c, \gamma) \circ G(b, \beta) = G (G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle, \beta\gamma).
\end{aligned}$$

Theorem 6.1. *Let $A \sharp B$ be an E -sum of A and B w.r.t. a function quintuple $(F, [-, -], G, H, \langle -, - \rangle)$. Then $A \sharp B$ is a near-ring if and only if the function quintuple is an amicable system for A w.r.t. B . In such a case, $A \sharp B$ is an extension of A by B .*

PROOF. Assume $A \sharp B$ is a near-ring. Since the addition is associative, it follows that

$$\begin{aligned}
(1) \quad & F(\alpha)(\beta) + [\alpha, \beta] + F(\alpha + \beta)(c) + [\alpha + \beta, \gamma] \\
& = F(\alpha) (b + F(\beta)(c) + [\beta, \gamma]) + [\alpha, \beta + \gamma]
\end{aligned}$$

From this equality and the initial conditions, (E1), (E2) and (E3) are obtained by putting $\beta = \gamma = 0$, $b = \gamma = 0$ and $b = c = 0$ respectively. The right distributivity of the multiplication over the addition, using (E1), gives

$$\begin{aligned}
(2) \quad & G(c, \gamma) (a + F(\alpha)(b) + [\alpha, \beta]) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma \rangle \\
& = G(c, \gamma)(a) + H(\alpha, \gamma)(c) + \langle \alpha, \gamma \rangle + F(\alpha\gamma) (G(c, \gamma)(b)) \\
& \quad + F(\alpha\gamma) (H(\beta, \gamma)(c)) + F(\alpha\gamma)(\langle \beta, \gamma \rangle) + [\alpha\gamma, \beta\gamma]
\end{aligned}$$

Substituting $\alpha = \beta = 0$ yields (E4). Then using this in (2) gives (E5). Using (E4), the associativity of the multiplication gives

$$\begin{aligned}
(3) \quad & G(c, \gamma) (G(b, \beta)(a)) + G(c, \gamma) (H(\alpha, \beta)(b)) + G(c, \gamma) (\langle \alpha, \beta \rangle) \\
& \quad + H(\alpha\beta, \gamma)(c) + \langle \alpha\beta, \gamma \rangle \\
& = G (G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle, \beta\gamma) (a) \\
& \quad + H(\alpha, \beta\gamma) (G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle) + \langle \alpha, \beta\gamma \rangle
\end{aligned}$$

Substituting $a = 0$ and $\alpha = 0$ in this equality, (E6) and (E7) respectively are obtained.

Conversely, if the function quintuple is an amicable system, we verify that $A \sharp B$ is a near-ring. The associativity of the addition will follow if the equality in (1) holds. Using (E1), (E3) and then (E2), the right hand side of (1) becomes:

$$\begin{aligned}
& F(\alpha) (b + F(\beta)(c) + [\beta, \gamma]) + [\alpha, \beta + \gamma] \\
& = F(\alpha)(b) + F(\alpha) (F(\beta)(c)) + F(\alpha) ([\beta, \gamma]) + [\alpha, \beta + \gamma] \\
& = F(\alpha)(b) + F(\alpha) (F(\beta)(c)) + [\alpha, \beta] + [\alpha + \beta, \gamma] \\
& = F(\alpha)(b) + [\alpha, \beta] + F(\alpha + \beta)(c) + [\alpha + \beta, \gamma].
\end{aligned}$$

It can be verified that $(0, 0)$ is the additive identity and every element (a, α) has an additive inverse $-(a, \alpha) = (-[-\alpha, \alpha] - F(-\alpha)(a), -\alpha)$. Hence $A \sharp B$ is a group. For the right distributivity, we need the equality in (2). Using E(4), the left hand side becomes

$$G(c, \gamma)(a) + G(c, \gamma) (F(\alpha)(b)) + G(c, \gamma) ([\alpha, \beta]) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma \rangle$$

which equals the right hand side in view of (E5).

Finally, for the associativity, we require the equality in (3). Using (E7) and (E6), the right hand side becomes

$$\begin{aligned} & G(c, \gamma) (G(b, \beta)(a)) + G(c, \gamma)(H(\alpha, \beta)(b)) + G(c, \gamma) (\langle \alpha, \beta \rangle) \\ & + H(\alpha\beta, \gamma)(c) + \langle \alpha\beta, \gamma \rangle = G (G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle, \beta\gamma) (a) \\ & + H(\alpha, \beta\gamma) (G(c, \gamma)(b) + H(\beta, \gamma)(c) + \langle \beta, \gamma \rangle) + \langle \alpha, \beta\gamma \rangle. \end{aligned}$$

Thus $A \sharp B$ is a near-ring. If we identify A with $(A, 0)$ and B with $\{(0, \alpha) + (A, 0) \mid \alpha \in B\}$, we have $A \triangleleft A \sharp B$ and $A \sharp B / A = B$.

Remark. Conditions (E1), (E2) and (E3), which are equivalent to $A \sharp B$ being a group under the induced addition, implies $F(\alpha) \in \text{Aut}(A^+)$ for all $\alpha \in B$. Indeed, by (E1) we only have to verify that $F(\alpha)$ is bijective. If $F(\alpha)(a) = F(\alpha)(b)$, then

$$(0, \alpha) + (a, 0) = (F(\alpha)(a), \alpha) = (F(\alpha)(b), \alpha) = (0, \alpha) + (b, 0).$$

Hence $(a, 0) = (b, 0)$ which yields the injectivity. Substituting $\beta = 0$ in (E2) gives $F(\alpha)(c) = F(\alpha) (F(0)(c))$; hence $c = F(0)(c)$ which yields $F(0) = 1_A$. Using (E3) with $\beta = -\alpha$ and $\gamma = \alpha$, gives $[\alpha, -\alpha] = F(\alpha) ([-\alpha, \alpha])$. Thus, for any $c \in A$, using (E2) with $\beta = -\alpha$ and (E1) yield $F(\alpha) ([-\alpha, \alpha]) + c = F(\alpha) (F(-\alpha)(c)) + F(\alpha) ([-\alpha, \alpha])$, i.e.

$$c = F(\alpha) (-[-\alpha, \alpha] + F(-\alpha)(c) + [-\alpha, \alpha])$$

which shows that $F(\alpha)$ is surjective.

All extensions of A by B are, up to equivalence, an E -sum of A and B for a suitable amicable system. This is our next result.

Theorem 6.2. *Let A and B be near-rings and let N be an extension of A by B . Then N is equivalent to an E -sum $A \sharp B$ for some amicable system $(F, [-, -], G, H, \langle -, - \rangle)$.*

PROOF. Each $\alpha \in B = N/A$ is a subset; let f be a choice function with $f(\alpha) \in \alpha$ and $f(0) = 0$. Every element $n \in N$ can uniquely be expressed as $n = a + f(\alpha)$ for some $a \in A$, $\alpha \in B$. Define a function

$\phi : N \rightarrow A \times B$ by $\phi (a + f(\alpha)) = (a, \alpha)$. Then ϕ is an injection. Define a quintuple of functions by:

$$\begin{aligned} F : B \rightarrow M(A), \quad & F(\alpha)(a) = f(\alpha) + a - f(\alpha) \\ [-, -] : B \times B \rightarrow A, \quad & [\alpha, \beta] = f(\alpha) + f(\beta) - f(\alpha + \beta) \\ G : A \times B \rightarrow A, \quad & G(b, \beta)(a) = a (b + f(\beta)) \\ H : B \times B \rightarrow M(A) \quad & \text{by } H(\alpha, \beta)(a) = f(\alpha)[a + f(\beta)] - f(\alpha)f(\beta) \\ \langle -, - \rangle : B \times B \rightarrow A \quad & \text{by } \langle \alpha, \beta \rangle = f(\alpha)f(\beta) - f(\alpha\beta). \end{aligned}$$

These functions are all well-defined; for example, we verify it for H : Since $f(\alpha) \in \alpha$ and $f(\beta) \in \beta$, $f(\alpha) = n_1 + a_1$ and $f(\beta) = n_2 + a_2$ for suitable $n_1, n_2 \in N$, $a_1, a_2 \in A$. Then

$$f(\alpha) [a + f(\beta)] - f(\alpha)f(\beta) = (n_1 + a_2) [a + (n_1 + a_2)] - (n_1 + a_1)(n_2 + a_2)$$

which is in A since $A \triangleleft N$. The quintuple $(F, [-, -], G, H, \langle -, - \rangle)$ satisfies the initial conditions since $f(0) = 0$. In addition, we will show that they form an amicable system for A w.r.t. B . But this will follow if we can show that $A \sharp B$ is a near-ring respect to the addition and multiplication induced by this function quintuple. To this effect, it is sufficient to show that ϕ preserves addition and multiplication. Firstly we note that

$$\begin{aligned} (a + f(\alpha)) + (b + f(\beta)) &= a + f(\alpha) + b - f(\alpha) + f(\alpha) + f(\beta) \\ &\quad - f(\alpha + \beta) + f(\alpha + \beta) = a + F(\alpha)(b) + [\alpha, \beta] + f(\alpha + \beta). \end{aligned}$$

The first three terms are in A ; hence it is the unique expression of $(a + f(\alpha)) + (b + f(\beta)) \in N$ in the form $c + f(\gamma)$. Thus

$$\begin{aligned} \phi \left((a + f(\alpha)) + (b + f(\beta)) \right) &= (a + F(\alpha)(b) + [\alpha, \beta], \alpha + \beta) \\ &= (a, \alpha) + (b, \beta). \end{aligned}$$

Likewise,

$$\begin{aligned} (a + f(\alpha)) (b + f(\beta)) &= a (b + f(\beta)) + f(\alpha) (b + f(\beta)) - f(\alpha)f(\beta) \\ &\quad + f(\alpha)f(\beta) - f(\alpha\beta) + f(\alpha\beta) \\ &= G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle + f(\alpha\beta) \end{aligned}$$

and the first three are terms in A . Hence

$$\begin{aligned} \phi \left((a + f(\alpha)) (b + f(\beta)) \right) &= (G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle, \alpha\beta) \\ &= (a, \alpha)(b, \beta). \end{aligned}$$

Hence ϕ is a near-ring isomorphism. In fact, it is an equivalence isomorphism: If $a \in A$, then $\phi(a) = \phi(a + 0) = (a, 0)$. As usual, we identify

$\{n + A \mid n \in N\}$ with $A \sharp B / (A, 0) = \{ (0, \alpha) + (A, 0) \mid \alpha \in B \}$ via $n = a + f(\alpha)$ for some unique $a \in A$, $\alpha \in B$. Then

$$\phi(n) + (A, 0) = (a, \alpha) + (A, 0) = (0, \alpha) + (A, 0) = n + A.$$

Remark. For any two near-rings A and B there always exists at least one E -sum $A \sharp B$ with an amicable system of functions, namely $F(\alpha) = 1_A$, $[\alpha, \beta] = 0 = \langle \alpha, \beta \rangle$, $G(c, \gamma)(a) = ac$ and $H(\alpha, \beta) = 0$. This is nothing but the direct sum $A \oplus B$ of the near-rings A and B .

An amicable system $(F, [-, -], G, H, \langle -, - \rangle)$ for A w.r.t. B is called a *factor-free amicable system* if $[\alpha, \beta] = 0 = \langle \alpha, \beta \rangle$. In such a case, it will be denoted by (F, G, H) and the initial conditions and the conditions (E1) to (E7) simplify to:

$F : B \rightarrow \text{Aut}(A^+)$ is a group homomorphism (i.e. $F(\alpha + \beta) = F(\alpha) \circ F(\beta)$).

$G : A \times B \rightarrow \text{End}(A^+)$ and

$H : B \times B \rightarrow M_0(A)$ are functions with $H(0, \alpha) = 0$ and

$$\begin{aligned} (F1) \quad & G(c, \gamma) (F(\alpha)(b)) + H(\alpha + \beta, \gamma)(c) \\ & = H(\alpha, \gamma)(c) + F(\alpha\gamma) (G(c, \gamma)(b)) + F(\alpha\gamma) (H(\beta, \gamma)(c)) \end{aligned}$$

$$\begin{aligned} (F2) \quad & G(c, \gamma) (H(\alpha, \beta)(b)) + H(\alpha\beta, \gamma)(c) \\ & = H(\alpha, \beta\gamma) (G(c, \gamma)(b) + H(\beta, \gamma)(c)) \end{aligned}$$

$$(F3) \quad G(c, \gamma) \circ G(b, \beta) = G (G(c, \gamma)(b) + H(\beta, \gamma)(c), \beta\gamma) .$$

((F1), (F2) and (F3) follows from (E5), (E6) and (E7) respectively.) A triple (f, g, h) , where $f, g, h \in M_0(A)$, is called a *triple homothetism* of A if $f = F(\alpha)$, $g = G(b, \beta)$ and $h = H(\beta, \alpha)$ for some $b \in A$, $\alpha, \beta \in B$ where (F, G, H) is a factorfree amicable system for A with respect to some B . If $I \triangleleft A$, then I is *invariant under the triple homothetism* (f, g, h) if $f(I) = I$, $g(I) \subseteq I$ and $h(I) \subseteq I$.

Theorem 6.3. *Let $\mathcal{M}$ be a class of near-rings. Then $\mathcal{M}$ satisfies condition (H) if and only every ideal $I \triangleleft A$ for $A \in \mathcal{M}$ is invariant under every triple homothetism of A .*

PROOF. Let $I \triangleleft A \in \mathcal{M}$, $\mathcal{M}$ satisfies condition (H), and let (f, g, h) be a triple homothetism of A . By definition, there is a near-ring B and a factorfree amicable system (F, G, H) for A w.r.t. B such that $f = F(\alpha_0)$, $g = G(b_0, \beta_0)$ and $h = H(\beta_0, \alpha_0)$. The E -sum $A \sharp B$ is a near-ring, w.r.t. the operations induced by F , G and H , and $(I, 0) \triangleleft (A, 0) \triangleleft A \sharp B$. By condition (H), $(I, 0) \triangleleft A \sharp B$; hence $(0, \alpha) + (i, 0) - (0, \alpha) \in (I, 0)$. This means $F(\alpha)(i) \in I$ and in particular $f(i) = F(\alpha_0)(i) \in I$. Thus $f(I) \subseteq I$.

For the reverse inclusion, since $F(\alpha)(i) \in I$ for all α , we have $i = F(\alpha_0)(F(-\alpha_0)(i)) \in F(\alpha_0)(I) = f(I)$; hence $f(I) = I$. Furthermore, $(i, 0)(b, \beta) \in (I, 0)$; hence $G(b, \beta)(i) \in I$ and in particular, $g(i) = G(b_0, \beta_0)(i) \in I$. Thus $g(I) \subseteq I$. Lastly, from $(0, \beta)[(0, \alpha) + (i, 0)] - (0, \beta)(0, \alpha) \in (I, 0)$ we have $H(\beta, \alpha)(F(\alpha)(i)) \in I$. In particular for $\beta = \beta_0$ and $\alpha = \alpha_0$ and since the restriction of $F(\alpha_0)$ to I is an automorphism of I , $h(i) = H(\beta_0, \alpha_0)(F(\alpha_0)(j)) \in I$ for a suitable $j \in I$. Thus $h(I) \subseteq I$.

Conversely, suppose the ideals of $A \in \mathcal{M}$ are invariant under triple homothetism of A . Consider $I \triangleleft A \triangleleft B$. Define functions $F : B \rightarrow \text{Aut}(A^+)$, $G : A \times B \rightarrow \text{End}(A^+)$ and $H : B \times B \rightarrow M_0(A)$ by $F(\alpha)(a) = \alpha + a - \alpha$, $G(a, \beta)(c) = c(a + \beta)$ and $H(\alpha, \beta)(a) = \alpha(a + \beta) - \alpha\beta$. Then (F, G, H) constitutes a factorfree amicable system for A w.r.t. B . Since I is invariant under triple homothetisms and $(F(\alpha), G(b, \beta), H(\beta, \alpha))$ is a triple homothetism for all $b \in A$, $\alpha, \beta \in B$, it follows that $I \triangleleft B$.

In conclusion we may mention that in the ring case, it is possible to express a double homothetism (in this case, $F(\alpha) = 1_A$ for all α and is thus omitted in the triple (f, g, h)) only in terms of A and the conditions (E1) to (E7) simplify considerably.

References

- [1] G. BETSCH and K. KAARLI, Supernilpotent radicals and non-hereditary classes of near-rings, *Coll. Math. Soc. J. Bolyai*, **38** (1985), 47–58, Radical Theory (Eger, 1982), *North Holland*.
- [2] L. RÉDEI, Algebra, vol. I, *Pergamon Press, Oxford*, 1967.
- [3] A. D. SANDS, On ideals in over-rings, *Publ. Math. Debrecen* **35** (1988), 274–279.
- [4] G. STEINEGGER, Erweiterungstheorie von Fastringen, *Doctoral Dissertation, Univ. Salzburg, Austria*, 1972.
- [5] S. VELDSMAN, Supernilpotent radicals of near-rings, *Comm. Algebra* **15** (1987), 2497–2509.
- [6] S. VELDSMAN, On the non-hereditariness of radical and semisimple classes of near-rings, *Studia Sci. Math. Hungar.* **24** (1989), 315–323.
- [7] S. VELDSMAN, Near-ring radicals with hereditary semisimple classes, *Archiv. Math.* **54** (1990), 443–447.
- [8] S. VELDSMAN, Extensions and ideals of Rings, *Publ. Math. Debrecen* **38** (1991), 297–309.
- [9] S. VELDSMAN, An overnilpotent radical theory for near-rings, *J. Algebra* **144** (1991), 248–265.

STEFAN VELDSMAN
DEPT. OF MATHEMATICS
UNIVERSITY OF PORT ELIZABETH
PO BOX 1600
PORT ELIZABETH
6000 SOUTH AFRICA

(Received June 11, 1990)