

On an application of the Zincenko method to the approximation of implicit functions

By IOANNIS K. ARGYROS (Lawton)

Abstract. Abstract We use the Zincenko iteration to approximate implicit functions in Banach spaces. The nonlinear equations involved contain a nondifferentiable term. Our hypotheses are more general than ZABREJKO-NGUEN's [10], in this case.

I. Introduction

Let E, Λ be Banach spaces and denote by $U(x_0, R)$ the closed ball with center $x_0 \in E$ and of radius R in E . We will use the same symbol for the norm $\| \cdot \|$ in both spaces. Suppose that the nonlinear operators $F(x, \lambda)$ and $G(x, \lambda)$ with values in E defined for $x \in U(x_0, R)$ and $\lambda \in U(\lambda_0, S)$ are such that F is Frechet differentiable there, $F'(x_0, \lambda_0)^{-1}$ exists and

- (1) $\| F'(x_0, \lambda_0)^{-1}(F'(x, \lambda) - F'(y, \lambda)) \| \leq K_1(r, s) \| x - y \|$,
- (2) $\| F'(x_0, \lambda_0)^{-1}(F'(x_0, \lambda) - F'(x_0, \lambda_0)) \| \leq K_2(s) \| \lambda - \lambda_0 \|$,
- (3) $\| F'(x_0, \lambda_0)^{-1}(G(x, \lambda) - G(y, \lambda)) \| \leq K_3(r, s) \| x - y \|$,

for all $x, y \in U(x_0, r) \subset U(x_0, R)$ and $\lambda \in U(\lambda_0, s) \subset U(\lambda_0, S)$. Here K_1, K_2 , and K_3 denote non-decreasing functions on the intervals $[0, R] \times [0, S]$, $[0, R]$ and $[0, R] \times [0, S]$ respectively.

We use the Zincenko iteration [11]

$$(4) \quad x_{n+1}(\lambda) = x_n(\lambda) - F'(x_n(\lambda), \lambda)^{-1}(F(x_n(\lambda), \lambda) + G(x_n(\lambda), \lambda)), n \geq 0$$

to approximate a solution $x^*(\lambda)$ of the equation

$$(5) \quad F(x, \lambda) + G(x, \lambda) = 0.$$

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By x_0 we mean $x_0(\lambda)$. That is x_0 depends on the λ used in (4).

Our assumptions (1)–(3) generalize the ones made by ZABREJKO–NGUEN [10], YAMAMOTO [9] and POTRA–PTÄK (in [6] (for $G = 0$)). Moreover, several authors have treated the case when $G = 0$ provided that K_1 and K_2 are constants (or not) [1], [2], [4], [5], [6].

We provide sufficient conditions for the convergence of iteration (4) to a locally unique solution $x^*(\lambda)$ of equation (5) as well as several error bounds on the distances $\|x_{n+1}(\lambda) - x_n(\lambda)\|$ and $\|x_n(\lambda) - x^*(\lambda)\|$.

We need to define the functions

$$a_s = K(s) \|F'(x_0, \lambda_0)^{-1}(F(x_0, \lambda) + G(x_0, \lambda))\|, \quad (s = 0 \text{ if } \lambda = \lambda_0),$$

$$w_s(r) = \int_0^r K_1(t, s) dt, \quad K_4(s) = \int_0^s K_2(t) dt, \quad k(s) = (1 - K_4(s))^{-1}$$

provided that

$$K_4(S) < 1, \quad \varphi_s(r) = a_s + K(s) \int_0^r w_s(t) dt - r,$$

$$\psi_s(r) = K(s) \int_0^r K_3(t, s) dt, \quad \chi_s(r) = \varphi_s(r) + \varphi_s(r),$$

and the iteration

$$(6) \quad \begin{aligned} y_{n+1}(\lambda) &= y_n(\lambda) - F'(x_0, \lambda_0)^{-1}(F(y_n(\lambda), \lambda) + G(y_n(\lambda), \lambda)), \\ y_0 &= x_0, \quad n \geq 0. \end{aligned}$$

II. Convergence results

We can now formulate the following result:

Theorem 1. *Suppose that the function $\chi_s(r)$ has a unique zero $\rho^* = \rho_s^*$ in $[0, R]$, and $\chi_s(R) \leq 0$. Then*

- (a) *equation (5) has a unique solution $x^*(\lambda) \in U(x_0, R)$ with $x^*(\lambda) \in U(x_0, \rho^*)$;*
- (b) *the following estimates are true*

$$(7) \quad \|y_{n+1}(\lambda) - y_n(\lambda)\| \leq v_{n+1} - v_n$$

and

$$(8) \quad \|y_n(\lambda) - x^*(\lambda)\| \leq \rho^* - v_n$$

where the scalar sequence $\{v_n\}$, $n \geq 0$ is monotonically increasing and convergent to ρ^* with

$$(9) \quad \begin{aligned} v_{n+1} &= d_s(v_n), \quad n \geq 0, \quad v_0 = 0 \\ d_s(r) &= r + \chi_s(r). \end{aligned}$$

PROOF. It is simple calculus to show that the sequence $\{v_n\}$, $n \geq 0$ is monotonically increasing and convergent to ρ^* (see also, [10, v. 675]). We will show using induction on n that the estimate (7) is true, from which (8) will follow immediately.

From (6) for $n = 0$ we get

$$\|y_1(\lambda) - y_0\| = \|F'(x_0, \lambda_0)^{-1}(F(x_0, \lambda) + G(x_0, \lambda))\| \leq a_s = d_s(0) = v_1 - v_0.$$

That is, the estimate (7) is true for $n = 0$. Let us assume that (7) is true for $n < k$. Then by (6), (1), (3), [10, p. 674] and the induction hypothesis we get

$$\begin{aligned} \|y_{k+1}(\lambda) - y_k(\lambda)\| &\leq \|y_k(\lambda) - y_{k-1}(\lambda) - F'(x_0, \lambda_0)^{-1}(F(y_k(\lambda), \lambda) - \\ &\quad - F(y_{k-1}(\lambda), \lambda))\| + \|F'(x_0, \lambda_0)^{-1}(G(y_k(\lambda), \lambda) - G(y_{k-1}(\lambda), \lambda))\| \leq \\ &\leq \int_0^1 \|F'(x_0, \lambda_0)^{-1}(F'((1-t)y_{k-1}(\lambda) + ty_k(\lambda)) - F'(x_0, \lambda_0))\| \cdot \\ &\quad \cdot \|y_k(\lambda) - y_{k-1}(\lambda)\| dt + \|F'(x_0, \lambda_0)^{-1}(G(y_k(\lambda), \lambda) - G(y_{k-1}(\lambda), \lambda))\| \leq \\ &\leq \int_0^1 w_s((1-t)v_{k-1} + tv_k)(v_k - v_{k-1}) dt + \int_{v_{k-1}}^{v_k} K_3(t, s) dt \leq \\ &\leq K(s) \left[\int_{v_{k-1}}^{v_k} w_s(t) dt + \int_{v_{k-1}}^{v_k} K_3(t, s) dt \right] = \\ &= d_s(v_k) - d_s(v_{k-1}) = v_{k+1} - v_k. \end{aligned}$$

That is, the estimate (7) is true for $n = k$. Hence the sequence $\{y_n(\lambda)\}$ is a Cauchy sequence in a Banach space and as such it converges to some $x^*(\lambda) \in U(x_0, \rho^*) \subset U(x_0, R)$. By letting $n \rightarrow \infty$ in (6) we deduce that $x^*(\lambda)$ is a solution of equation (5). We will show that $x^*(\lambda)$ is the unique solution of equation (5) in $U(x_0, R)$, by considering the sequences

$$(10) \quad \begin{aligned} z_{n+1}(\lambda) &= z_n(\lambda) - F'(x_0, \lambda_0)^{-1}(F(z_n(\lambda), \lambda) + G(z_n(\lambda), \lambda)), \\ z_0 &\in U(x_0, R), \quad n \geq 0 \end{aligned}$$

and

$$(11) \quad w_{n+1} = d_s(w_n), \quad n \geq 0, \quad w_0 = R.$$

It is enough to show

$$(12) \quad \| y_n(\lambda) - z_n(\lambda) \| \leq w_n - v_n, \quad n \geq 0.$$

It is simple calculus to show that the scalar sequence given by (11) is monotonically convergent to ρ^* . Hence, if for z_0 we choose the second solution $y^*(\lambda) \in U(x_0, r)$ of equation (5) then by (12)

$$\| x^*(\lambda) - y^*(\lambda) \| \leq w_n - v_n.$$

That is, $x^*(\lambda) = y^*(\lambda)$.

For $n = 0$, (12) becomes $\| y - z_0 \| \leq R - 0 = R$. Hence, (12) is true for $n = 0$. Let us assume that (12) holds for $n \leq k$ then by (6), (10) as before we get

$$\begin{aligned} & \| y_{k+1}(\lambda) - z_{k+1}(\lambda) \| \leq \| z_k(\lambda) - y_k(\lambda) - F'(x_0, \lambda_0)^{-1}(F(z_k(\lambda), \lambda) - \\ & - F(y_k(\lambda), \lambda)) \| + \| F'(x_0, \lambda_0)^{-1}(G(z_k(\lambda), \lambda) - G(y_k(\lambda), \lambda)) \| \leq \\ & \leq \int_0^1 \| F'(x_0, \lambda_0)^{-1}(F'((1-t)y_k(\lambda) + tz_k(\lambda)) - F'(x_0, \lambda_0)) \| \cdot \\ & \cdot \| z_k(\lambda) - y_k(\lambda) \| dt + \int_{v_k}^{w_k} K_3(t, s) dt \leq \int_0^1 w_s((1-t)v_k + tw_k) \\ & (w_k - v_k) dt + \int_{v_k}^{w_k} K_3(t, s) dt \leq K(s) \left[\int_{v_k}^{w_k} w_s(t) dt + \right. \\ & \left. + \int_{v_k}^{w_k} K_3(t, s) dt \right] = d_s(w_k) - d_s(v_k) = w_{k+1} - v_{k+1}. \end{aligned}$$

That completes the proof of the theorem.

We can now formulate the main result:

Theorem 2. *Suppose that the hypotheses of Theorem 1 are true. Then*

(a) *the sequence $\{\rho_n\}, n \geq 0$ given by*

$$\rho_{n+1} = \rho_n + u_s(\rho_n), \quad n \geq 0, \quad \rho_0 = 0 \text{ with } u_s(r) = -\frac{\chi_s(r)}{\varphi'_s(r)}$$

is monotonically increasing and converges to ρ^ .*

(b) *The iterates generated by (4) are well defined for all $n \geq 0$ and remain in $U(x_0, \rho^*)$.*

(c) *Moreover the following estimates are true*

$$(13) \quad \| x_{n+1}(\lambda) - x_n(\lambda) \| \leq \rho_{n+1} - \rho_n, \quad n \geq 0$$

and

$$(14) \quad \|x_n(\lambda) - x^*(\lambda)\| \leq \rho^* - \rho_n, \quad n \geq 0.$$

PROOF. Part (a) can be shown exactly as in Proposition 3 in [10, p. 677]. We will only show (13), since (14) will follow then from it immediately. For $n = 0$ we get $\|x_1(\lambda) - x_0\| \leq a_s = \rho_1 - \rho_0$. That is, (13) is true for $n = 0$. Let us assume that (13) is true for $n < k$. By the induction hypothesis

$$\|x_k(\lambda) - x_0\| \leq \sum_{j=1}^k \|x_j(\lambda) - x_{j-1}(\lambda)\| \leq \sum_{j=1}^k (\rho_j - \rho_{j-1}) = \rho_k,$$

the Banach lemma on invertible operators, (2) and the estimate

$$\begin{aligned} \|F'(x_0, \lambda_0)^{-1}(F'(x_k(\lambda), \lambda) - F'(x_0, \lambda_0))\| &\leq \\ &\leq K(s)w_s(\rho_k) < K(s)w_s(\rho^*) = \rho'_s(\rho^*) + 1 \leq 1, \end{aligned}$$

it follows that $F'(x, \lambda)$ is invertible for all $\lambda \in U(\lambda_0, S)$, $x \in U(x_0, R)$ and

$$\begin{aligned} &\|F'(x_k(\lambda), \lambda)^{-1}F'(x_0, \lambda_0)\| \leq \\ &\leq \| [I + F'(x_0, \lambda)^{-1}(F'(x, \lambda) - F'(x_0, \lambda_0))]^{-1} \| \cdot \\ &\cdot \|F'(x_0, \lambda)^{-1}F'(x_0, \lambda_0)\| \leq -\frac{K(s)}{\varphi'_s(\rho_k)}. \end{aligned}$$

Then by (4), (1)–(3), (15) and the induction hypothesis we get

$$\begin{aligned} &\|x_{k+1}(\lambda) - x_k(\lambda)\| = \|F'(x_k(\lambda), \lambda)^{-1}(F(x_k(\lambda), \lambda) + G(x_k(\lambda), \lambda))\| = \\ &= \|F'(x_k(\lambda), \lambda)^{-1}(F(x_k(\lambda), \lambda) - F(x_{k-1}(\lambda), \lambda) - F'(x_{k-1}(\lambda), \lambda)(x_k(\lambda) - \\ &\quad - x_{k-1}(\lambda) + G(x_k(\lambda), \lambda) - G(x_{k-1}(\lambda), \lambda)))\| \leq \\ &\leq F'(x_k(\lambda), \lambda)^{-1}F'(x_0, \lambda_0) \left\| \int_0^1 \|F'(x_0, \lambda_0)^{-1}(F'((1-t)x_{k-1}(\lambda) + \right. \\ &\quad \left. + tx_k(\lambda)) - F'(x_{k-1}(\lambda))\| \cdot \|x_k(\lambda) - x_{k-1}(\lambda)\| dt + \right. \\ &\quad \left. + \|F'(x_0, \lambda_0)^{-1}(G(x_k(\lambda), \lambda)) - G(x_{k-1}(\lambda), \lambda))\| \right\| \leq \\ &\leq -\frac{K(s)}{\varphi'_s(\rho_k)} \int_0^1 (w_s((1-t)\rho_{k-1} + t\rho_k) - w_s(\rho_{k-1}))(\rho_k - \rho_{k-1}) dt - \\ &\quad - \frac{1}{\varphi'_s(\rho_k)} (\psi_s(\rho_k) - \psi_s(\rho_{k-1})) \leq \end{aligned}$$

$$\begin{aligned}
&\leq - \frac{\varphi_s(\rho_k) - \varphi_s(\rho_{k-1}) - \varphi'_s(\rho_{k-1})(\rho_k - \rho_{k-1}) + \psi_s(\rho_{k-1}) - \psi_s(\rho_{k-1})}{\varphi'_s(\rho_k)} = \\
&= \rho_{k+1} - \rho_k.
\end{aligned}$$

Hence (13) is true for $n = k$. That completes the proof of the theorem.

We will now derive some a posteriori error bounds for iteration (4).
Let $r_{n,s} = r_n = \|x_n(\lambda) - x_0\|$,

$$q_{n,s}(r) = q_n(r) = K_1(r_n + r, s), \quad f_{n,s}(r) = f_n(r) = K_3(r_n + r, s)$$

for $r \in [0, R - r_n]$ and set

$$a_{n,s} = a_n = \|x_{n+1}(\lambda) - x_n(\lambda)\|, \quad b_{n,s} = b_n = K(s)(1 - K(s)w_s(r_n))^{-1}.$$

Without loss of generality we assume than $a_n > 0$.

Then exactly as in Theorem 2 in [9, p. 989] we can show

Theorem 3. *Suppose that the hypotheses of Theorem 1 are satisfied.
Then*

(a) *the equation*

$$r = a_n + b_n \int_0^r (r - t)q_n(t) + f_n(t) dt$$

has a unique positive zero $\rho_{n,s}^ = \rho_n^*$ in the interval $[0, R - r_n]$,
 $n \geq 0$ and $\rho_0^* = \rho^*$.*

(b) *The following estimates are true:*

$$\begin{aligned}
(16) \quad &\|x_n(\lambda) - x^*(\lambda)\| \leq \rho_n^* \\
&\leq (\rho^* - \rho_n)a_n/\Delta\rho_n, \quad n \geq 0, \\
&\leq (\rho^* - \rho_n)a_{n-1}/\Delta\rho_{n-1}, \quad n \geq 1, \\
&\leq \rho^* - \rho_n, \quad n \geq 0,
\end{aligned}$$

where $\Delta\rho_n = \rho_{n+1} - \rho_n$.

That is, our bound (16) is sharper than Miel-type bounds [3], [7] and more general than the corresponding one in [9, p. 989].

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IOANNIS K. ARGYROS
DEPARTMENT OF MATHEMATICAL SCIENCES
CAMERON UNIVERSITY
LAWTON
OK 73505
U.S.A.

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