

Two remarks on Hosszú's functional inequality

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Hosszú's functional inequality

$$(1) \quad f(x+y-xy) + f(xy) \leq f(x) + f(y)$$

was considered in [1] and the following results were proved:

- (i) If $f : (0, 1) \rightarrow R$ is a concave function, then f satisfies (1);
- (ii) if $a \in [23/16, 3/2)$ is fixed then the function f defined on $(0, 1)$ by $f(x) = -x^4 + 2x^3 - ax^2$ is a continuous solution of (1) and f is not concave.

Here, we shall give two generalizations of (i). We say $f : I \rightarrow R$ ($I = (0, 1)$) is a Wright-convex function if for each $y > x$ and $\delta > 0$ ($y + \delta, x \in I$),

$$(2) \quad f(x + \delta) - f(x) \leq f(y + \delta) - f(y).$$

f is Wright-concave if the reverse inequality holds in (2). If C is a class of convex functions, and W a class of Wright-convex functions, then $C \subset W$, the inclusion being proper.

Using the substitutions: $x \rightarrow xy$, and $\delta \rightarrow x - xy$, we get:

Theorem 1. *If $f : (0, 1) \rightarrow R$ is a Wright-concave function, then f satisfies (1).*

Note that the above proof is shorter than that from [1]. Now, we shall give a multidimensional generalization of Theorem 1.

Let R^k denote the k -dimensional vector lattice of points $X = (x_1, \dots, x_k)$, x_i real for $i = 1, \dots, k$, with the partial ordering

$$X = (x_1, \dots, x_k) \leq Y = (y_1, \dots, y_k)$$

if and only if $x_i \leq y_i$ for $i = 1, \dots, k$. We shall write

$$aX + bY = (ax_1 + by_1, \dots, ax_k + by_k), \text{ where } a, b \in R, X, Y \in R^k,$$

and

$$XY = (x_1y_1, \dots, x_ky_k).$$

A real-valued function f on an interval $I = (0, 1)^k$ will be said to have nondecreasing increments if

$$(3) \quad f(A + H) - f(A) \leq f(B + H) - f(B)$$

whenever $A \in I$, $B + H \in I$, $0 = (0, \dots, 0) \leq H \in R^k$, $A \leq B$.

For some properties of these functions see [2]. Of course, if the reverse inequality in (3) holds then f is a function with nonincreasing increments.

Using the substitutions: $A = XY$, $B = Y$, $H = X - XY$, we get

Theorem 2. *If $f : (0, 1)^k \rightarrow R$ is a function with nonincreasing increments, then*

$$(4) \quad f(X + Y - XY) + f(XY) \leq f(X) + f(Y).$$

In fact, using the Jensen inequality for these functions we have

$$f(X + Y - XY) + f(XY) \leq f(X) + f(Y) \leq 2f((X + Y)/2).$$

References

- [1] GY. MAKSA and ZS. PÁLES, On Hosszú's functional inequality, *Publicationes Math.* **36** (1989), 187–189.
- [2] H. D. BRUNK, Integral inequalities for functions with nondecreasing increments, *Pacific J. Math.* **14** (1964), 783–793.

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