

On compositions of distributions

By KOU HUAIZHONG (Xixing City) and BRIAN FISHER (Leicester)

Abstract. Using a double neutrix limit, the paper investigates the compositions of one-dimensional distributions and obtains a universal definition of the compositions of distributions.

1. Introduction

Since 1950, much investigation has been carried out on the compositions of distributions, see [3], [6] and [7], but only in the case where the functions involved have been continuous or at most locally summable. No meaning could be given to expressions of the form $H(\delta^{(s)}(x))$, $\delta^{(r)}(\delta^{(s)}(x))$, $[\delta^{(r)}(x)]^s$ and the like. The purpose of this paper is to achieve a new universal definition of the compositions of distributions, where a double neutrix limit is used. Accordingly, the questions not resolved are worked out.

2. Preliminaries

2.1. Neutrix and Neutrix Limit

The following definition of a neutrix was given by van der CORPUT [1]:

Definition 2.1. Let N' be a set and let N be a commutative, additive group of functions mapping N' into a commutative, additive group N'' . If N has the property that the only constant function in N is the zero function, then N is said to be a neutrix and the functions in N are said to be negligible.

Now suppose that N' is a subspace of a topological space X having a limit point y which is not contained in N' . Let N'' be the real (or complex)

numbers and let N be a commutative, additive group of functions mapping N' into N'' with the property that if N contains a function $\nu(x)$ which converges to a finite limit c as x tends to y , then $c = 0$. N is a neutrix, since if f is in N and $f(x) = c$ for all x in N' , then $f(x)$ converges to the finite limit c as x tends to y and so $c = 0$.

This leads us to a second definition of van der CORPUT [1].

Definition 2.2. Let $f(x)$ be a real (or complex) valued function defined on N' and suppose it is possible to find a constant c such that $f(x) - c$ is negligible in N . Then c is called the neutrix limit of $f(x)$ as x tends to y and we write

$$N-\lim_{x \rightarrow y} f(x) = c.$$

Note that if a neutrix limit exists, then it is unique, since if $f(x) - c$ and $f(x) - c'$ are in N , then the constant function $c - c'$ is also in N and so $c = c'$.

In the following we let N the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$, range the real numbers and $y = \infty$, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We will use n or m to denote a general term in N' so that if $\{a_n\}$ is a sequence of real numbers, then $N-\lim_{n \rightarrow \infty} a_n$ means exactly the same thing as $N-\lim_{m \rightarrow \infty} a_m$.

Note that if $\{a_n\}$ is a sequence of real numbers which converges to a in the normal sense as n tends to infinity, then the sequence $\{a_n\}$ converges to a in the neutrix sense as n tends to infinity and

$$\lim_{n \rightarrow \infty} a_n = N-\lim_{n \rightarrow \infty} a_n.$$

2.2. Convolution and regularity

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,

$$(iv) \int_{-1}^1 \rho(x) dx = 1.$$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

3. Compositions

Since 1983, the second author in [3], [4] and [5] has investigated the composition of distributions using neutrix limits. The following definition was given in [5] and is the most general.

Definition 3.1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to $h(x)$ on the interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \phi(x) dx = \langle h(x), \phi(x) \rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b) , where

$$F_n(x) = (F * \delta_n)(x)$$

for $n = 1, 2, \dots$ and N is the neutrix given in the previous section.

We now give an alternative definition for the distribution $F(f)$.

Definition 3.2. Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $F(f(x))$ exists and is equal to $h(x)$ on the interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f_m(x)) \phi(x) dx \right] = \langle h(x), \phi(x) \rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b) , where

$$F_n(x) = (F * \delta_n)(x), \quad f_m(x) = (f * \delta_m)(x)$$

for $m, n = 1, 2, \dots$.

In the following theorem, we show that Definition 3.2 generalizes Definition 3.1 for bounded, locally summable functions f .

Theorem 3.1. *Let F be a distribution in $\mathcal{D}$ and let f be a bounded, locally summable function. If the distribution $F(f(x))$ exists and equals $h(x)$ under Definition 3.1 on the interval (a, b) , then $F(f(x))$ also exists under Definition 3.2 on the interval (a, b) and is equal to $h(x)$.*

PROOF. Suppose that $F(f(x))$ exists and equals $h(x)$ under Definition 3.1 on the interval (a, b) . Then

$$N\text{-}\lim_{n \rightarrow \infty} \langle F_n(f(x)), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all ϕ in $\mathcal{D}(a, b)$. Now

$$\lim_{m \rightarrow \infty} \int_I |f_m(x) - f(x)| dx = 0,$$

for any bounded interval I , since f is a bounded, summable function. Further, since F_n is a continuously differentiable function and f, f_n are bounded, it follows that

$$|F_n(f_m(x)) - F_n(f(x))| \leq K_n |f_m(x) - f(x)|,$$

for some K_n . We therefore have

$$\left| \int_a^b [F_n(f_m(x)) - F_n(f(x))] \phi(x) dx \right| \leq MK_n \int_c^d |f_m(x) - f(x)| dx,$$

where

$$M = \sup\{|\phi(x)|\}$$

and $[c, d]$ is a bounded interval containing the support of ϕ and so

$$\lim_{m \rightarrow \infty} \int_a^b F_n(f_m(x)) \phi(x) dx = \int_a^b F_n(f(x)) dx,$$

or equivalently

$$N\text{-}\lim_{m \rightarrow \infty} \langle F_n(f_m(x)), \phi(x) \rangle = \langle F_n(f(x)), \phi(x) \rangle.$$

Thus

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle F_n(f_m(x)), \phi(x) \rangle \right] &= N\text{-}\lim_{n \rightarrow \infty} \langle F_n(f(x)), \phi(x) \rangle \\ &= \langle F(f(x)), \phi(x) \rangle \\ &= \langle h(x), \phi(x) \rangle, \end{aligned}$$

and it follows that $F(f(x))$ exists and equals $h(x)$ by Definition 3.2.

It is an open question as to whether Definition 3.2 is a generalization of Definition 3.1 for all locally summable functions.

From now on, the compositions that we will consider will be using Definition 3.2.

Theorem 3.2. *Let F be a bounded, continuous, summable function on the real line. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = F(0),$$

for $s = 0, 1, 2, \dots$.

PROOF. We put

$$F_n(x) = (F * \delta_n)(x), \quad \delta_m^{(s)}(x) = (\delta^{(s)} * \delta_m)(x)$$

for $m, n = 1, 2, \dots$.

Choosing an arbitrary $\varepsilon > 0$, there exists an M such that $m\varepsilon > 1$ for $m > M$. Then with $m > M$, we have

$$\delta_m^{(s)}(x) = m^{s+1} \rho(mx) = 0$$

for $|x| > \varepsilon$ and so

$$F_n(\delta_m^{(s)}(x)) = F_n(0)$$

for $|x| > \varepsilon$. Thus, for arbitrary ϕ in $\mathcal{D}$ with support contained in the interval (a, b) , which we may suppose contains the origin,

$$\begin{aligned} \langle F_n(\delta_m^{(s)}(x)), \phi(x) \rangle &= \int_a^b F_n(\delta_m^{(s)}(x)) \phi(x) dx \\ &= F_n(0) \int_a^{-\varepsilon} \phi(x) dx + F_n(0) \int_{\varepsilon}^b \phi(x) dx + \int_{-\varepsilon}^{\varepsilon} F_n(\delta_m^{(s)}(x)) \phi(x) dx, \end{aligned}$$

for $m > M$. Thus

$$\begin{aligned} |\langle F_n(\delta_m^{(s)}(x)) - F_n(0), \phi(x) \rangle| &= \left| \int_{-\varepsilon}^{\varepsilon} [F_n(\delta_m^{(s)}(x)) - F_n(0)] \phi(x) dx \right| \\ &\leq 2\varepsilon K \end{aligned}$$

for $m > M$, where

$$K = \sup \left\{ |[F_n(\delta_m^{(s)}(x)) - F_n(0)] \phi(x)| : m, n = 1, 2, \dots; x \in \mathbf{R} \right\} < \infty,$$

since F_n and ϕ are bounded functions. It follows that

$$\lim_{m \rightarrow \infty} \langle F_n(\delta_m^{(s)}(x)), \phi(x) \rangle = \langle F_n(0), \phi(x) \rangle$$

and so

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle F_n(\delta_m^{(s)}(x)), \phi(x) \rangle \right] &= \lim_{n \rightarrow \infty} \langle F_n(0), \phi(x) \rangle \\ &= \langle F(0), \phi(x) \rangle. \end{aligned}$$

This proves that $F(\delta^{(s)}(x))$ exists and is equal to $F(0)$ for $s = 0, 1, 2, \dots$.

Theorem 3.3. *The distribution $H(\delta^{(s)}(x))$ exists on the real line and*

$$H(\delta^{(s)}(x)) = \frac{1}{2},$$

for $s = 0, 1, 2, \dots$, where H denotes Heaviside's function.

PROOF. We put

$$H_n(x) = (H * \delta_n)(x)$$

for $n = 1, 2, \dots$, so that

$$\begin{aligned} H_n(x) &= \begin{cases} 1, & x > 1/n, \\ \int_{-1/n}^x \delta_n(t) dt, & |x| \leq 1/n, \\ 0, & x < -1/n, \end{cases} \\ 0 \leq H_n(x) &\leq 1, \quad H_n(0) = \frac{1}{2}, \end{aligned}$$

for $n = 1, 2, \dots$.

Choosing arbitrary $\varepsilon > 0$, there exists M such that $m\varepsilon > 1$ for $m > M$. It then follows as above that

$$H_n(\delta_m^{(s)}(x)) = H_n(0) = \frac{1}{2},$$

for $|x| > \varepsilon$ and $m > M$ and so

$$\left| \left\langle H_n(\delta_m^{(s)}(x)) - \frac{1}{2}, \phi(x) \right\rangle \right| = \left| \int_{-\varepsilon}^{\varepsilon} \left[H_n(\delta_m^{(s)}(x)) - \frac{1}{2} \right] \phi(x) dx \right| \leq \int_{-\varepsilon}^{\varepsilon} |\phi(x)| dx$$

for $m > M$ and arbitrary ϕ in $\mathcal{D}$. The result of the theorem follows as above.

Theorem 3.4. *Let F be a bounded, summable function on the real line which is continuous everywhere except for a simple discontinuity at the origin. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = \frac{1}{2}[F(0+) + F(0-)]$$

for $s = 0, 1, 2, \dots$.

PROOF. Let $F(0+) - F(0-) = c$. Then the function G defined by

$$G(x) = F(x) - cH(x)$$

satisfies the conditions of Theorem 3.2. Thus

$$G(\delta^{(s)}(x)) = G(0) = F(0-),$$

and so

$$\begin{aligned} G(\delta^{(s)}(x)) + cH(\delta^{(s)}(x)) &= F(0-) + \frac{1}{2}[F(0+) - F(0-)] \\ &= \frac{1}{2}[F(0+) + F(0-)] \end{aligned}$$

for $s = 0, 1, 2, \dots$. The result of the theorem follows.

Theorem 3.5. *The distribution $\delta^{(r)}(\delta^{(s)}(x))$ exists on the real line and*

$$\delta^{(r)}(\delta^{(s)}(x)) = 0$$

for $r, s = 0, 1, 2, \dots$.

PROOF. Choosing arbitrary $\varepsilon > 0$, there exists M such that $m\varepsilon > 1$ for $m > M$. Then $m\varepsilon > 1$ and $|x| > \varepsilon$ implies that $\rho^{(s)}(mx) = 0$ and so

$$\begin{aligned} \delta_n^{(r)}(\delta_m^{(s)}(x)) &= n^{r+1} \rho^{(r)}(nm^{s+1} \rho^{(s)}(mx)) \\ &= n^{r+1} \rho^{(r)}(0) \end{aligned}$$

for $|x| > \varepsilon$ and $m > M$.

Thus, for $m > M$ and all ϕ in $\mathcal{D}$, we have

$$\begin{aligned} \langle \delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x) \rangle &= n^{r+1} \rho^{(r)}(0) \int_{|x| > 1/m} \phi(x) dx + \\ &+ n^{r+1} \int_{|x| < 1/m} \rho^{(r)}(nm^{s+1}(mx)) \phi(x) dx \rightarrow n^{r+1} \rho^{(r)}(0) \int_{-\infty}^{\infty} \phi(x) dx \end{aligned}$$

as m tends to infinity. It follows that

$$\lim_{m \rightarrow \infty} \langle \delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x) \rangle = n^{r+1} \langle \rho^{(r)}(0), \phi(x) \rangle$$

and so

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle \delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x) \rangle \right] &= N\text{-}\lim_{n \rightarrow \infty} n^{r+1} \langle \rho^{(r)}(0), \phi(x) \rangle \\ &= 0. \end{aligned}$$

The result of the theorem follows. This completes the proof of the theorem.

Theorem 3.6. *The distribution $[\delta^{(r)}(x)]^s$ exists on the real line and*

$$(1) \quad [\delta^{(r)}(x)]^s = \frac{(-1)^{rs+s-1} c(\rho, r, s)}{(rs+s-1)!} \delta^{(rs+s-1)}(x)$$

for $r = 0, 1, 2, \dots$ and $s = 2, 3, \dots$, where

$$c(\rho, r, s) = \int_{-1}^1 [\rho^{(r)}(y)]^s y^{rs+s-1} dy.$$

In particular

$$(2) \quad [\delta^{(r)}(x)]^s = 0$$

for even s .

PROOF. Put

$$\begin{aligned} (x^s)_n &= x^s * \delta_n(x) = \int_{-1/n}^{1/n} (x-t)^s \delta_n(t) dt \\ &= \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} x^i \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt \end{aligned}$$

for $n = 1, 2, \dots$, where

$$\binom{s}{i} = \frac{s!}{i!(s-i)!}.$$

Then

$$(3) \quad [(\delta_m^{(r)}(x))^s]_n = \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} [\delta_m^{(r)}(x)]^i \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt,$$

where

$$\delta_m^{(r)}(x) = m^{r+1} \rho^{(r)}(mx),$$

the support of $\delta_m^{(r)}$ being contained in the interval $[-1/m, 1/m]$. Making the substitution $y = mx$ we have

$$\int_{-1/m}^{1/m} [\delta_m^{(r)}(x)]^i x^j dx = m^{ri+i-j-1} \int_{-1}^1 [\rho^{(r)}(y)]^i y^j dy.$$

It follows that

$$(4) \quad N\text{-}\lim_{m \rightarrow \infty} \int_{-1/m}^{1/m} [\delta_m^{(r)}(x)]^i x^j dx = 0,$$

for $i = 0, 1, \dots, s$, $j = 0, 1, 2, \dots$ and $j \neq ri + i - 1$.

In the particular case $j = ri + i - 1$ we have

$$(5) \quad \int_{-\infty}^{\infty} [\delta_m^{(r)}(x)]^i x^{ri+i-1} dx = \int_{-1}^1 [\rho^{(r)}(y)]^i y^{ri+i-1} dy = c(\rho, r, i).$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then by Taylor's theorem we have

$$\phi(x) = \sum_{j=0}^{rs+s-1} \frac{\phi^{(j)}(0)}{j!} x^j + \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} x^{rs+s},$$

where $0 \leq \xi \leq 1$. Thus

$$\begin{aligned} \int_{-1/m}^{1/m} [(\delta_m^{(r)}(x))^i]_n \phi(x) dx &= \sum_{j=0}^{rs+s-1} \frac{\phi^{(j)}(0)}{j!} \int_{-1/m}^{1/m} [\delta_m^{(r)}(x)]^i x^j dx \\ &\quad + \int_{-1/m}^{1/m} \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} [\delta_m^{(r)}(x)]^i x^{rs+s} dx, \end{aligned}$$

where

$$\begin{aligned} \left| \int_{-1/m}^{1/m} \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} [\delta_m^{(r)}(x)]^i x^{rs+s} dx \right| &\leq \\ &\leq \frac{2m^{(r+1)(i-s)-1}}{(rs+s)!} \sup\{|\phi^{(rs+s)}(x)|\} \cdot \sup\{|\rho^{(r)}(x)|\} \\ &\rightarrow 0 \end{aligned}$$

as m tends to infinity for $i = 0, 1, \dots, s$.

Using equations (4) and (5), it follows that

$$N\text{-}\lim_{m \rightarrow \infty} \int_{-1/m}^{1/m} [(\delta_m^{(r)})^i]_n \phi(x) dx = \begin{cases} 0, & i = 0, \\ \frac{c(\rho, r, i) \phi^{(ri+i-1)}(0)}{(ri+i-1)!}, & i = 1, \dots, s. \end{cases}$$

It now follows from equation (3) that

$$\begin{aligned} \langle [(\delta_m^{(r)}(x))^s]_n, \phi(x) \rangle &= \\ &= \sum_{i=0}^s (-1)^i \binom{s}{i} \int_{-1/m}^{1/m} [(\delta_m^{(r)}(x))^{s-i}]_n \phi(x) dx \cdot \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt \end{aligned}$$

and it follows from what we have just proved that

$$\begin{aligned} N\text{-}\lim_{m \rightarrow \infty} \langle [(\delta_m^{(r)}(x))^s]_n, \phi(x) \rangle &= \\ &= \sum_{i=1}^s (-1)^{s-i} \binom{s}{i} \frac{c(\rho, r, i) \phi^{(ri+i-1)}(0)}{(ri+i-1)!} \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt, \end{aligned}$$

where

$$\int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt = \begin{cases} n^{i-s} \int_{-1}^1 u^{s-i} \rho(u) du, & i = 1, \dots, s-1 \\ 1, & i = s, \end{cases}$$

Thus

$$\begin{aligned} N-\lim_{n \rightarrow \infty} \left[N-\lim_{m \rightarrow \infty} \langle [(\delta_m^{(r)}(x))^s]_n, \phi(x) \rangle \right] &= \frac{c(\rho, r, s) \phi^{(rs+s-1)}(0)}{(rs+s-1)!} \\ &= \frac{(-1)^{rs+s-1} c(\rho, r, s)}{(rs+s-1)!} \langle \delta^{(rs+s-1)}(x), \phi(x) \rangle \end{aligned}$$

and equation (1) follows. Equation (2) follows on noticing that

$$[\rho^{(r)}(y)]^s y^{rs+s-1}$$

is an odd function for even s and so $c(\rho, r, s) = 0$ for even s . This completes the proof of the theorem.

The next definition for the product of two distributions was given in [2].

Definition 3.3. Let f and g be distributions in D' and let

$$f_n(x) = (f * \delta_n)(x), \quad g_n(x) = (g * \delta_n)(x).$$

Then the product $f.g$ is defined to exist and be equal to the distribution h on the interval (a, b) if

$$N-\lim_{n \rightarrow \infty} \langle f_n(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b) .

We note that with this definition of the product of two distributions, the definition of the distribution f^2 as the composition of the function x^2 and the distribution f if it exists, is distinct from the definition of the product $f.f$ if it exists. However, the following theorem holds:

Theorem 3.7. *Let f be a distribution in D' . Then the distribution f^2 exists on the interval (a, b) if and only if the distribution $f.f$ exists on the interval (a, b) and then*

$$f^2 = f \cdot f$$

on the interval (a, b) .

PROOF. It follows as in the proof of Theorem 3.6 that

$$[(f_m(x))^2]_n = \int_{-1/n}^{1/n} t^2 \delta_n(t) dt - 2f_m(x) \int_{-1/n}^{1/n} t \delta_n(t) dt + [f_m(x)]^2 \int_{-1/n}^{1/n} \delta_n(t) dt,$$

where

$$\lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} t^2 \delta_n(t) dt = \lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} t \delta_n(t) dt = 0, \quad \int_{-1/n}^{1/n} \delta_n(t) dt = 1.$$

Then it follows that f^2 exists on the interval (a, b) , if and only if

$$N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle [f_m(x)]^2 \rangle_n, \phi(x) \right]$$

exists and is equal to

$$N\text{-}\lim_{m \rightarrow \infty} \langle [f_m(x)]^2 \rangle_n, \phi(x) = N\text{-}\lim_{n \rightarrow \infty} \langle f_n(x) f_n(x), \phi(x) \rangle,$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b) . That is, if and only if $f.f$ exists on the interval (a, b) .

References

- [1] J. G. VAN DER CORPUT, Introduction to the neutrix calculus, *J. Analyse Math.* **7** (1959–60), 291–398.
- [2] B. FISHER, The neutrix distribution product $x_+^{-r} \delta^{(r-1)}(x)$, *Studia. Sci. Math. Hungar.* **9** (1974), 439–441.
- [3] B. FISHER, On defining the distribution $\delta^{(r)}(f(x))$, *Rostock. Math. Kolloq.* **23** (1983), 73–80.
- [4] B. FISHER, On defining the change of variable in distributions, *Rostock. Math. Kolloq.* **28** (1985), 75–86.
- [5] B. FISHER, On defining the distribution $\delta^{(r)}(f(x))$ for summable f , *Publ. Math. Debrecen* **32** (1985), 233–241.
- [6] A. ŁOJASIEWICZ, Sur la valeur et la limite d'une distribution en un point, *Studia Math.* **16** (1957), 1–36.
- [7] J. MIKUSIŃSKI AND R. SIKORSKI, The elementary theory of distributions, *Dissertationes Math. Rozprawy Mat.* **12** (1957).

KOU HUAI ZHONG
DEPARTMENT OF MATHEMATICS
HENAN NORMAL UNIVERSITY
XIXING CITY, HENAN PROVINCE
PEOPLE'S REPUBLIC OF CHINA

BRIAN FISHER
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY, LEICESTER
LE1 7RH
ENGLAND

(Received 24 July, 1990)