

Weak automorphisms of the permutation groups S_n

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0. The notion of weak automorphisms was studied in [3] and [6]. Let G be a group and let us denote by $A^{(n)}$ the class of all n -ary words in G , i.e. the set of all functions $f : G^n \rightarrow G$ of the form

$$(1) \quad f(x_1, x_2, \dots, x_n) = x_{i_1}^{m_1} \cdot x_{i_2}^{m_2} \cdot \dots \cdot x_{i_l}^{m_l},$$

where $m_1, m_2, \dots, m_l \in \mathbb{Z}$ (= integers), $i_1, i_2, \dots, i_l \in \{1, 2, \dots, n\}$, and $n = 1, 2, \dots$. We call a permutation τ of the set G a weak automorphism of the group G if the mapping $\tau^* : A^{(n)} \rightarrow A^{(n)}$ defined by the formula:

$$(2) \quad (\tau^* f)(x_1, x_2, \dots, x_n) = \tau f(\tau^{-1}(x_1), \tau^{-1}(x_2), \dots, \tau^{-1}(x_n))$$

is a bijection, where $n = 1, 2, \dots, n$. We denote the set of all weak automorphisms by $\text{Aut}^* G$. (cf. [1], [2])

Observe that e is the unique element of $A^{(0)}$ and therefore we have $\tau(e) = e$ for all $\tau \in \text{Aut}^* G$ (cf. [2]).

The purpose of the paper is to prove the following Theorem

Theorem 0. *Each weak automorphism τ of the permutation group S_n is of the form $\tau(x) = \alpha(x)^m$, where α is an automorphism of the group S_n and m is a positive integer $< \exp S_n$, coprime to $n!$, and the representation is unique.*

1. In [1] the following proposition is shown:

Proposition 1.1. *For any group G , the group $\text{Aut} G$ is a normal subgroup of $\text{Aut}^* G$.*

Let us start with the simple consequence of the Proposition 1.1.

Lemma 1.1. *Let G be a group, $x, y \in G$ and $\tau \in \text{Aut}^* G$. The following conditions are equivalent:*

- a) *there exists an automorphism φ of G such that $\varphi(x) = y$;*
- b) *there exists an automorphism ψ of G such that $\psi(\tau(x)) = \tau(y)$.*

Lemma 1.2. *Let H be a set of generators of a group G . If $\tau \in \text{Aut}^* G$, $\beta \in \text{Aut } G$ and $\tau(x) = x$ and $\beta(x) \in H$ for all $x \in H$, then $\tau \circ \beta = \beta \circ \tau$.*

PROOF. By proposition 1.1 $\tau^{-1} \circ \beta^{-1} \circ \tau \circ \beta \in \text{Aut } G$, and it is the identity on the generating set H , so it is the identity automorphism. $\square$

For any word f of the form (1) we denote by $S^i(f)$ the sum of exponents of x_i .

Lemma 1.3. *If τ is a weak automorphism of a group G and f is a word of the form $f(x_1, \dots, x_n) = x_1 \dots x_n$, then $x = x^{S^i(\tau^* f)}$ for any element $x \in G$ and $i = 1, 2, \dots, n$.*

PROOF. $x = \tau f(e, \dots, e, \tau^{-1}(x), e, \dots, e) = x^{S^i(\tau^* f)}$.

Theorem 1.1. *If τ is a weak automorphism of a group G then $\tau(x^n) = \tau(x)^n$ for all $n \in \mathbb{Z}$.*

PROOF. For $n = 0$ Theorem 1.1 means that $\tau(e) = e$.

Now we prove Theorem 1.1 for $n \geq 1$. Let $f(x_1, x_2, \dots, x_n) = x_1 \cdot x_2 \cdot \dots \cdot x_n$. From Lemma 1.3, we have:

$$\tau(x^n) = \tau^*(f(\tau(x), \dots, \tau(x))) = \tau(x)^{S^1(\tau^* f)} \dots \tau(x)^{S^n(\tau^* f)} = \tau(x)^n.$$

Applying Lemma 1.3 in the case $n = 2$ for the weak automorphism τ^{-1} we get:

$$x = x^{S^1((\tau^{-1})^* f)} = x^{S^2((\tau^{-1})^* f)}.$$

Hence

$$\tau(x) \cdot \tau(x^{-1}) = f(\tau(x), \tau(x^{-1})) = \tau(((\tau^{-1})^* f)(x, x^{-1})) = \tau(xx^{-1}) = e$$

and Theorem 1.1 follows. $\square$

Corollary 1.1. *Each weak automorphism of any group preserves the order of its element.*

The following Lemma 1.4 is a generalization of Lemma 1.3.

Lemma 1.4. *If τ is a weak automorphism of a group G and f is of the form (1), then*

$$x^{S^i f} = x^{S^i(\tau^* f)}$$

for $i = 1, 2, \dots, n$ and $x \in G$.

PROOF. It follows from Theorem 1.1. $\square$

Lemma 1.5. *Let S be a subset of a group G with the property $x \in S$ implies $x^n \in S$ for all integers n . If $x, y \in S$ and $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$ are integers such that:*

$$x^{a_1+a_2+\dots+a_k} = y^{b_1+b_2+\dots+b_k} = e,$$

where $k = 1, 2, \dots$, then:

$$x^{a_1}y^{b_1} \dots x^{a_k}y^{b_k} \in [S, S] = \text{gp} \{s_1^{-1}s_2^{-1}s_1s_2 : s_1, s_2 \in S\}.$$

PROOF. This is obvious for $k = 1$.

Let us suppose that we have proved Lemma 1.5 for $k \leq t$. To complete the proof it is enough to notice that we have:

$$\begin{aligned} & x^{a_1} \cdot y^{b_1} \dots x^{a_{t+1}} \cdot y^{b_{t+1}} = \\ & = x^{a_1}y^{b_1} \dots x^{a_{t-1}}y^{b_{t-1}}x^{a_t+a_{t+1}}y^{b_t+b_{t+1}} [y^{b_t+b_{t+1}}, x^{a_{t+1}}] [x^{a_{t+1}}, y^{b_{t+1}}]. \quad \square \end{aligned}$$

Theorem 1.2. *Let S be a subset of a group G with the property $x \in S$ implies $x^n \in S$ for all $n = 1, 2, \dots$. Then $\tau([S, S]) = [\tau(S), \tau(S)]$ for every weak automorphism τ of the group G .*

PROOF. At first we prove that $\tau([S, S]) \subset [\tau(S), \tau(S)]$. Let τ be a weak automorphism of a group G and $f(x_1, x_2) = [x_1, x_2]$. We know from Theorem 1.1 that $x \in \tau(S)$ implies $x^n \in \tau(S)$. Since τ is a weak automorphism, $(\tau^*f)(u, v) = u^{a_1}v^{b_1} \dots u^{a_k}v^{b_k}$ for some integers $a_1, \dots, a_k, b_1, \dots, b_k$. Hence, applying Lemma 1.4 and Lemma 1.5, we get $\tau([x, y]) = \tau f(x, y) = (\tau^*f)(\tau(x), \tau(y)) \in [\tau(S), \tau(S)]$.

To complete the proof of Theorem 1.2 it is enough to consider τ^{-1} instead of τ and $\tau(S)$ instead of S . $\square$

Corollary 1.2. *We have $[x, y] = e$ if and only if $[\tau(x), \tau(y)] = e$ for any weak automorphism τ of a group G , and $x, y \in G$.*

Theorem 1.3. *Suppose that τ is a weak automorphism of a group G and x, y are elements of G such that $[x, y] = e$. Then we have $[\tau(x), \tau(y)] = e$ and $\tau(xy) = \tau(x)\tau(y)$.*

PROOF. Let $[x, y] = e$ and $f(x, y) = x \cdot y$. By Corollary 1.2, $[\tau(x), \tau(y)] = e$. Moreover, $\tau(xy) = \tau^*f(\tau(x), \tau(y)) = \tau(x)^{S^1(\tau^*f)} \cdot \tau(y)^{S^2(\tau^*f)} = \tau(x)\tau(y)$ according to Lemma 1.3, which completes the proof. $\square$

Corollary 1.3. *Every weak automorphism of an abelian group is in fact an automorphism.*

2. Theorem 2.1. *Let G be a group of finite exponent n .*

- a) *The mapping $x \rightarrow x^k$ is a bijection iff $(k, n) = 1$.*
- b) *The inverse of $x \rightarrow x^k$ is $x \rightarrow x^m$, where $k \cdot m \equiv 1 \pmod n$.*
- c) *If $(k, n) = 1$ then $x \rightarrow x^k$ is a weak automorphism of G .*

PROOF. a) and b): If $x \rightarrow x^k$ is a bijection then for every $x \in G$ we have $(|x|, k) = 1$. So, $(k, n) = 1$. Conversely, if $(k, n) = 1$ then there exists m such that $k \cdot m \equiv 1 \pmod n$. Therefore, the mapping $x \rightarrow x^k$ is a bijection, where $x \rightarrow x^m$ is the inverse mapping.

c) Let $\tau(x) = x^k$, where $(k, n) = 1$, and let f be a n -ary word of the form

$$f(x_1, x_2, \dots, x_n) = x_{i_1}^{m_1} \cdot x_{i_2}^{m_2} \cdot \dots \cdot x_{i_l}^{m_l}.$$

Then $\tau^{-1}(x) = x^m$ and so:

$$\begin{aligned} (\tau^* f)(x_1, x_2, \dots, x_n) &= \tau f(\tau^{-1}(x_1), \tau^{-1}(x_2), \dots, \tau^{-1}(x_n)) = \\ &= (x_{i_1}^{mm_1} \cdot x_{i_2}^{mm_2} \cdot \dots \cdot x_{i_l}^{mm_l})^k. \end{aligned}$$

Hence τ^* is a mapping $A^{(n)}$ into $A^{(n)}$. But $(\tau^{-1})^*$ is also a mapping $A^{(n)}$ into $A^{(n)}$ and $(\tau^{-1})^* \circ \tau^* = \text{id}|_{A^{(n)}}$. Hence τ^* is a bijection of $A^{(n)}$ and therefore τ is weak automorphism of G . $\square$

Theorem 2.2. *For any group of exponent n the set of all weak automorphisms of the form $x \rightarrow x^k$ is a subgroup of the center of $\text{Aut}^* G$ isomorphic to the multiplicative group Z_n^* of the ring Z_n .*

PROOF. It follows from Theorem 1.1 and Theorem 2.1. $\square$

Theorem 2.3. *For any group G of exponent n the set of all weak automorphisms of the form $x \rightarrow \alpha(x)^k$, where $\alpha \in \text{Aut} G$, is a normal subgroup of $\text{Aut}^* G$.*

PROOF. It is a simple consequence of Proposition 1.1, Theorem 2.1 and Theorem 2.2. $\square$

The following Example shows that the set of all weak automorphisms of the form $x \rightarrow \alpha(x)^k$, where $\alpha \in \text{Aut} G$, could be a proper normal subgroup of $\text{Aut}^* G$.

Example 2.1. Let G be a group of the form:

$$G = \langle a, b \mid [[a, b], a] = [[a, b], b] = [a, b]^4 = 1 \rangle.$$

Each element $g \in G$ is of the form

$$g = a^p b^q [a, b]^r,$$

where $r = 0, 1, 2, 3$, and this form is unique. The exponent of the group G is infinite, so, if a weak automorphism τ is of the form $\tau = \alpha(x)^n$, then

$n = 1$ or $n = -1$. Hence, each weak automorphism of the form $\tau = \alpha(x)^n$ satisfies one of the two equations:

$$\tau(x \cdot y) = \tau(x) \cdot \tau(y) \quad \text{or} \quad \tau(x \cdot y) = \tau(y) \cdot \tau(x).$$

Let τ be defined by

$$\tau(a^p b^q [a, b]^r) = a^p b^q [a, b]^{pq+3r}.$$

It is easy to check that:

- a) $\tau \circ \tau = \text{id}$,
- b) $\tau(x \cdot y) = \tau(x)\tau(y)[\tau(x), \tau(y)]$,
- c) $\tau(x \cdot y) \neq \tau(x) \cdot \tau(y)$ and $\tau(x \cdot y) \neq \tau(y) \cdot \tau(x)$.

Hence, the function τ is a weak automorphism and it is not of the form $\alpha(x)^n$. $\square$

3. Now we show that any weak automorphism of the group S_n of all permutations on n letters is of the form $x \rightarrow \alpha(x)^k$, where $\alpha \in \text{Aut } S_n$, $(k, |S_n|) = 1$.

Let $B_{k,n}$, $1 \leq k \leq n/2$, denote the set of all compositions of k disjoint transpositions in the group S_n .

Lemma 3.1. *Let τ be a weak automorphism of a group S_n , which satisfies the condition $\tau(B_{1,n}) = B_{1,n}$. If $\tau((i, j)) = (p, q)$ and $\tau((i, k)) = (p, r)$ then $\tau((j, k)) = (q, r)$.*

PROOF. Let us put $f(x, y) = x \cdot y \cdot x$. For $x = (i, j)$ and $y = (i, k)$ we have $\tau((j, k)) = \tau(x \cdot y \cdot x) = (\tau^* f)(\tau(x), \tau(y)) = (\tau^* f)((p, r), (p, q))$, which is a permutation on three letters p, q, r . It follows from our hypothesis that $\tau((j, k))$ is a transposition. But τ is a bijection and so $\tau((j, k)) = (p, q)$. $\square$

Lemma 3.2. *If τ is a weak automorphism of the permutation group S_n , which satisfies the condition $\tau(B_{1,n}) = B_{1,n}$, then there exists an inner automorphism α of S_n such that $\tau(x) = \alpha(x)$ for each transposition x .*

PROOF. Theorem 1.3 gives $[\tau((1, 2)), \tau((2, 3))] \neq e$. Therefore there exists $\sigma(1), \sigma(2), \sigma(3)$ such that $\tau((1, 2)) = (\sigma(1), \sigma(2))$ and $\tau((2, 3)) = (\sigma(2), \sigma(3))$. The equality $\tau((1, 3)) = (\sigma(1), \sigma(2))$ follows from Lemma 3.1.

Let us suppose that σ is a function: $\{1, \dots, k\} \rightarrow \{1, \dots, n\}$ such that $\tau((i, j)) = (\sigma(i), \sigma(j))$, where $i, j = 1, 2, \dots, k$, $k \geq 3$. Since τ is a bijection and $\tau((1, 2)) = (\sigma(1), \sigma(2))$ and $\tau((1, 3)) = (\sigma(1), \sigma(3))$, so $\tau((1, k+1)) = (\sigma(1), d)$, where d is not equal to $\sigma(1), \sigma(2), \dots, \sigma(k)$. Let us put $\sigma(k+1) = d$. The inductive conclusion follows from Lemma 3.1. So we have constructed a permutation $\sigma \in S_n$ such that $\tau((i, j)) = (\sigma(i), \sigma(j))$ for $i, j = 1, 2, \dots, n$. Hence we have $\tau|_{B_{1,n}} = \alpha|_{B_{1,n}}$, where $\alpha(x) = \sigma x \sigma^{-1}$ and Lemma 3.2 follows. $\square$

Lemma 3.3. *For every weak automorphism τ of S_n there exists an automorphism α of S_n such that $\alpha(x) = \tau(x)$ for all $x \in B_{1,n}$.*

PROOF. Let $n \neq 6$. Then it follows from Lemma 1.1 that for any weak automorphism τ of S_n there exists k such that $\tau(B_{1,n}) = B_{k,n}$. From [4] we know that $|B_{1,n}| \neq |B_{k,n}|$ for $n \neq 6$ and $k \neq 1$. Hence $\tau(B_{1,n}) = B_{1,n}$ and the conclusion follows from Lemma 3.2.

Now let $n = 6$. As we know from [5] for each automorphism β of S_6 $\beta(B_{2,6}) = B_{2,6}$. We have $|B_{1,6} \cup B_{3,6}| = 30$ and $|B_{2,6}| = 45$ so from Lemma 1.1 we obtain that $\tau(B_{2,6}) = B_{2,6}$.

Moreover, we shall show that $\tau(B_{1,6}) = B_{1,6}$ or $\tau(B_{1,6}) = B_{3,6}$. Indeed, if $\tau(B_{1,6}) \neq B_{1,6}$ and $\tau(B_{1,6}) \neq B_{3,6}$ then let us consider transpositions (i, j) and (j, k) such that $\tau((i, j)) \in B_{1,6}$ and $\tau((j, k)) \in B_{3,6}$. From Theorem 1.3 we know that $[\tau((i, j)), \tau((j, k))] \neq e$ so $\tau((i, j))$ and $\tau((j, k))$ are of the form $\tau((i, j)) = (p, q)$ and $\tau((j, k)) = (p, s)(q, t)(u, w)$. Then $\tau((i, j, k)) = \tau((i, j)(j, k)) = (\tau^*f)(\tau((i, j)), \tau((j, k))) = (\tau^*f)((p, q), (p, s)(q, t)(u, w))$, where $f(x, y) = x \cdot y$. We know that $(\tau^*f)((p, q), (p, s)(q, t)(u, w))$ is of the form $\mu \circ (u, w)$, where the permutation μ fixes u and w . Therefore its order is not equal to the order of (i, j, k) , which contradicts Corollary 1.1.

From [5] we also know that there exists an automorphism β of S_6 such that $\beta(B_{3,6}) = B_{1,6}$. Let β be an automorphism of S_6 such that $\beta(B_{3,6}) = B_{1,6}$ if $\tau(B_{1,6}) = B_{3,6}$ and $\beta = \text{id}$ if not. Then the conclusion follows from Lemma 3.2 after applying it for $\beta \circ \tau$. $\square$

In the proof of Lemma 3.5 we use the following obvious number-theoretical Lemma 3.4:

Lemma 3.4. *If $m_1, m_2, \dots, m_n$ are positive integers such that $m_{rt} \equiv m_r \pmod r$ for all $r \cdot t \leq n$, then there exists m such that $m \equiv m_k \pmod k$ for all $k \leq n$.*

Lemma 3.5. *If τ is a weak automorphism of S_n such that $\tau(x) = x$ for each transposition x , then there exists a positive integer m such that $\tau(y) = y^m$ for all $y \in S_n$.*

PROOF. First, we prove that if x is a permutation on some letters $i_1, i_2, \dots, i_k$ then $\tau(x)$ is a permutation on the same letters. Indeed, there exists transpositions $x_1, x_2, \dots, x_m$ on letters $i_1, i_2, \dots, i_k$ such that $x = x_1 \cdot x_2 \cdot \dots \cdot x_m$. Let us put $f(x_1, x_2, \dots, x_m) = x_1 \cdot x_2 \cdot \dots \cdot x_m$. Since $\tau(x_i) = x_i$ for each transposition x_i , so we have $\tau(x) = \tau f(x_1, x_2, \dots, x_m) = \tau^*(f)(\tau(x_1), \tau(x_2), \dots, \tau(x_m)) = \tau^*(f)(x_1, x_2, \dots, x_m)$.

Now we prove that if x is a cycle of the form $x = (i_1, i_2, \dots, i_k)$, then $\tau(x)$ is also a cycle on the same letters. Indeed, suppose that $\tau(x)$ is not a cycle. Then $\tau(x) = y_1 \cdot y_2$ for some independent permutations. From Theorem 1.3 we know that $x = \tau^{-1}(y_1 \cdot y_2) = \tau^{-1}(y_1) \cdot \tau^{-1}(y_2)$

and of course $\tau^{-1}(y_1)$ and $\tau^{-1}(y_2)$ are independent, which contradicts the assumption that x is a cycle.

Now we prove that for each cycle x we have $\tau(x) = x^m$ for some m (dependent on x) not greater than length of x . Indeed, applying Lemma 1.2 for the set H of all transpositions of S_n , for $\beta(y) = xyx^{-1}$ and for $y = x$ we get $\tau(x)x = x\tau(x)x^{-1} = x\tau(x)$. So $\tau(x) = x^m$, where m is a positive integer.

Further, let us consider two cycles x, y of length k . We have $\tau(x) = x^m$ and $\tau(y) = y^w$. There exists an inner automorphism β of S_n such that $\beta(x) = y$, therefore, using Lemma 1.2, we get $y^w = \tau\beta(x) = \beta\tau(x) = \beta(x)^m = y^m$; hence $m = w$. So, there exists positive integers $m_2, m_3, \dots, m_n$ such that $\tau(x) = x^{m_k}$ for each cycle x of length k . Moreover, if x is a composition of some disjoint cycles of length k then also $\tau(x) = x^{m_k}$, because of Theorem 1.3.

Now let r, t be positive integers such that $r, r \cdot t \leq n$ and let x be a cycle of length $r \cdot t$. Then x^t is a composition of disjoint cycles of length r . So, using Theorem 1.1, we have $x^{m_{rt}} = \tau(x^t)\tau(x)^t = x^{m_{tr}t}$. Hence $m_{tr} \equiv m_r \pmod{r}$. Using Lemma 3.4 we get a positive integer m such that $m \equiv m_k \pmod{k}$. Hence $\tau(x) = x^m$ for each cycle x of S_n .

If x is a composition of disjoint cycles of S_n then using Theorem 1.3 we conclude that also $\tau(x) = x^m$. So Lemma 3.5 is proved. $\square$

Now we will PROVE THEOREM 0 from the introduction, which is the main result of the paper.

PROOF. Let τ be a weak automorphism of the group S_n . As we know from Lemma 3.3, there exists an automorphism α of S_n such that $\alpha^{-1}\tau(x) = x$ for each transposition x of S_n . It follows from Lemma 3.5 that there exists a positive integer m such that $\alpha^{-1}\tau(x) = x^m$. Hence $\tau(x) = \alpha(x)^m$. Theorem 2.1 implies that m is coprime to $n!$. Since an automorphism β of S_n can not be of the form $\beta(x) = x^k$ for $k \neq 1$, therefore the representation $\tau(x) = \alpha(x)^m$, with $0 < m < \exp(S_n)$, is unique and the result follows. $\square$

Corollary 3.1. *The group Aut^*S_n is the direct product of the group $\text{Aut}S_n$ and the group H of weak automorphisms of the form $x \rightarrow x^k$, where k is coprime to $n!$, and H is isomorphic to the multiplication group Z_m^* of the ring Z_m , where $m = \exp S_n$.*

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