

On certain tensor fields on contact metric manifolds

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Dedicated to Professor Lajos Tamássy on his 70th birthday

In Sasakian manifolds, MATSUMOTO and CHŪMAN [3] defined a contact Bochner curvature tensor (see also YANO [6]). This is invariable with respect to a D-homothetic deformation (see TANNO [5] about a D-homothetic deformation). In this paper we define a new tensor field which is invariable with respect to the D-homothetic deformation in a contact metric manifold and call it an E-tensor. Then we shall study the contact metric manifold with vanishing E-tensor.

1. Preliminaries

Let M be a $(2n + 1)$ -dimensional contact metric manifold with the structure tensors (ϕ, ξ, η, g) , where ϕ is a linear mapping $TM \rightarrow TM$ (TM is the tangent bundle over M), ξ is a vector field, η a 1-form, and g a Riemannian metric on M , such that

$$\begin{aligned} \phi\xi = 0, \quad \eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(\xi, X) \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = d\eta(X, Y) \end{aligned}$$

for any vector fields X and Y on M . Now we define an operator h by $h = -\frac{1}{2}\mathcal{L}_\xi\phi$ where $\mathcal{L}$ denotes Lie differentiation. Then the vector field ξ is Killing if and only if h vanishes. It is well known that h and ϕh are symmetric operators, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\eta \circ h = 0$, $\text{Tr } h = 0$ and $\text{Tr } \phi h = 0$ (see [1]). Moreover the following general formulas for a contact metric manifold M were obtained

$$(1.1) \quad \nabla_X \xi = \phi X + \phi h X$$

$$(1.2) \quad g(Q\xi, \xi) = 2n - \text{Tr } h^2$$

$$(1.3) \quad \sum_{i=1}^{2n+1} (\nabla_{E_i} \phi) E_i = -2n\xi \quad (\{E_i\} \text{ is an orthonormal frame})$$

$$(1.4) \quad (\nabla_{\phi X} \phi) \phi Y + (\nabla_X \phi) Y = -2g(X, Y)\xi + \eta(Y)(X + hX + \eta(X)\xi) \\ \left((\nabla_X \phi)\xi = -\eta(X)\xi + X + hX \right)$$

$$(1.5) \quad \phi(\nabla_\xi h)X = X - \eta(X)\xi - h^2X - R(X, \xi)\xi,$$

where ∇ is the covariant differentiation with respect to g , Q is the Ricci operator of M , R is the curvature tensor field of M and $\text{Tr } h$ denotes the trace of h (see [1] [2] and [4]). Moreover, by means of $\phi h\xi = 0$, we get

$$(1.6) \quad \phi(\nabla_Y h)\xi = -hY - h^2Y.$$

If ξ is Killing in a contact metric manifold M , M is said to be a K -contact Riemannian manifold. If a contact metric manifold M is normal (i.e., $N + d\eta \otimes \xi = 0$, where N denotes the Nijenhuis tensor formed with ϕ), M is called a Sasakian manifold. A Sasakian manifold is a K -contact Riemannian manifold. In a Sasakian manifold with structure tensors (ϕ, ξ, η, g) we have

$$\nabla_X \xi = \phi X, \quad (\nabla_X \phi)Y = R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X \\ \phi Q = Q\phi, \quad Q\xi = 2n\xi \quad (\text{see e.g., [7]}).$$

2. D-homothetic deformations

Let M be an $(m+1)$ -dimensional ($m = 2n$) contact metric manifold. Now we define the tensor field $\tilde{B}$ in M by

$$(2.1) \quad \tilde{B}(X, Y) = R(X, Y) + \frac{1}{m+4} (QY \wedge X - QX \wedge Y \\ + Q\phi Y \wedge \phi X - Q\phi X \wedge \phi Y + g(Q\phi X, Y)\phi - g(Q\phi Y, X)\phi \\ + 2g(\phi X, Y)Q\phi + \eta(Y)QX \wedge \xi + \eta(X)\xi \wedge QY) \\ - \frac{k+m}{m+4} (\phi Y \wedge \phi X + 2g(\phi X, Y)\phi) - \frac{k-4}{m+4} Y \wedge X \\ + \frac{k}{m+4} (\eta(Y)\xi \wedge X + \eta(X)Y \wedge \xi),$$

where $k = \frac{S+m}{m+2}$ (S is the scalar curvature of M) and $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$ (cf., [3]). If M is a Sasakian manifold, $\tilde{B}$ coincides

with the contact Bochner curvature tensor by MATSUMOTO and CHŪMAN [3] and the following are satisfied

$$(2.2) \quad \begin{aligned} \tilde{B}(\xi, Y)Z &= \tilde{B}(X, Y)\xi = 0, & \eta(\tilde{B}(X, Y)Z) &= 0, \\ \tilde{B}(\phi X, \phi Y)Z &= \tilde{B}(X, Y)Z, \end{aligned}$$

where we have used $R(\phi X, Y)Z - R(\phi Y, X)Z = g(\phi Z, X)Y - g(\phi Z, Y)X - g(Z, X)\phi Y + g(Z, Y)\phi X$ in a Sasakian manifold.

We consider a D -homothetic deformation $g^* = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta$, $\phi^* = \phi$, $\xi^* = \alpha^{-1}\xi$, $\eta^* = \alpha\eta$ on a contact metric manifold M , where α is a positive constant, and for a D -homothetic deformation we say that $M(\phi, \xi, \eta, g)$ is D -homothetic to $M(\phi^*, \xi^*, \eta^*, g^*)$. It is well known that if a contact metric manifold $M(\phi, \xi, \eta, g)$ is D -homothetic to $M(\phi^*, \xi^*, \eta^*, g^*)$, $M(\phi^*, \xi^*, \eta^*, g^*)$ is a contact metric manifold. Moreover if $M(\phi, \xi, \eta, g)$ is a K -contact Riemannian manifold (resp. Sasakian manifold), then $M(\phi^*, \xi^*, \eta^*, g^*)$ is also a K -contact Riemannian manifold (resp. Sasakian manifold) (see [5]). Denoting by W^i_{jk} the difference $\Gamma^*_{jk}{}^i - \Gamma_{jk}{}^i$ of Christoffel symbols, by (1.1), we have in a contact metric manifold

$$\begin{aligned} W(X, Y) &= (\alpha - 1)(\eta(Y)\phi X + \eta(X)\phi Y) \\ &\quad + \frac{\alpha - 1}{2\alpha}((\nabla_X \eta)(Y) + (\nabla_Y \eta)(X))\xi \quad (\text{see [5]}) \\ &= (\alpha - 1)(\eta(Y)\phi X + \eta(X)\phi Y) + \frac{\alpha - 1}{\alpha}g(\phi hX, Y)\xi. \end{aligned}$$

Putting this into

$$\begin{aligned} R^*(X, Y)Z &= R(X, Y)Z + (\nabla_X W)(Z, Y) - (\nabla_Y W)(Z, X) \\ &\quad + W(W(Z, Y), X) - W(W(Z, X), Y) \end{aligned}$$

and using (1.1), we have

$$(2.3) \quad \begin{aligned} R^*(X, Y)Z &= R(X, Y)Z + (\alpha - 1) \\ &\quad \times \left(2g(\phi X, Y)\phi Z + g(\phi Z, Y)\phi X - g(\phi Z, X)\phi Y + \eta(Y)(\nabla_X \phi)Z \right. \\ &\quad \left. - \eta(X)(\nabla_Y \phi)Z + \eta(Z)(\nabla_X \phi)Y - \eta(Z)(\nabla_Y \phi)X \right) \\ &\quad - (\alpha - 1)^2 \left(\eta(Z)\eta(X)Y - \eta(Z)\eta(Y)X \right) \\ &\quad - \frac{\alpha - 1}{\alpha} \left(g(X, (\nabla_Y \phi)hZ)\xi - g(Y, (\nabla_X \phi)hZ)\xi + g(X, \phi(\nabla_Y h)Z)\xi \right. \\ &\quad \left. - g(Y, \phi(\nabla_X h)Z)\xi + g(X, \phi hZ)\phi hY - g(Y, \phi hZ)\phi hX \right) \end{aligned}$$

$$-\frac{(\alpha-1)^2}{\alpha} \left(\eta(X)g(hZ, Y)\xi - \eta(Y)g(hZ, X)\xi \right).$$

Here, choosing a ϕ -basis and using (1.3) and (1.5), we get

$$(2.4) \quad \begin{aligned} \text{Ric}^*(X, Y) &= \text{Ric}(X, Y) \\ &+ (\alpha-1) \left(-2g(X, Y) + 2(2n+1)\eta(X)\eta(Y) \right) \\ &+ 2n(\alpha-1)^2\eta(X)\eta(Y) - \frac{\alpha-1}{\alpha} \left(-g(X, Y) \right. \\ &\left. + \eta(X)\eta(Y) - 2g(hX, Y) + g(hX, hY) + g(R(X, \xi)\xi, Y) \right), \end{aligned}$$

where Ric is the Ricci curvature of M . From (2.4) we find

$$(2.5) \quad \begin{aligned} Q^*X &= \frac{1}{\alpha}QX + \frac{\alpha-1}{\alpha} \left(-2X + 2(2n+1)\eta(X)\xi \right) \\ &\quad - \frac{\alpha-1}{\alpha^2}g(X, Q\xi)\xi - 2n \left(\frac{\alpha-1}{\alpha} \right)^2 \eta(X)\xi \\ &\quad - \frac{\alpha-1}{\alpha^2} \left(-X + \eta(X)\xi - 2hX + h^2X + R(X, \xi)\xi \right), \end{aligned}$$

where we have used $Q^*\xi = \frac{1}{\alpha}Q\xi - \frac{\alpha-1}{\alpha^2}g(Q\xi, \xi)\xi + 2n\frac{\alpha^2-1}{\alpha^2}\xi$.

By virtue of (1.2) we have

$$(2.6) \quad S^* = \frac{1}{\alpha}S - 2n\frac{\alpha-1}{\alpha} + \frac{\alpha-1}{\alpha^2} \text{Tr } h^2.$$

From (2.1), (2.3), (2.4), (2.5) and (2.6), after some lengthy computation we obtain

$$(2.7) \quad \begin{aligned} \tilde{B}^*(X, Y)Z &= \tilde{B}(X, Y)Z + (\alpha-1) \left(\eta(Y)g(X, Z)\xi \right. \\ &\quad \left. - \eta(X)g(Y, Z)\xi + 2\eta(X)\eta(Z)Y - 2\eta(Y)\eta(Z)X + \eta(Y)(\nabla_X\phi)Z \right. \\ &\quad \left. + \eta(Z)(\nabla_X\phi)Y - \eta(X)(\nabla_Y\phi)Z - \eta(Z)(\nabla_Y\phi)X \right) \\ &\quad - \frac{\alpha-1}{\alpha} \left(g(X, (\nabla_Y\phi)hZ)\xi - g(Y, (\nabla_X\phi)hZ)\xi + g(X, \phi(\nabla_Yh)Z)\xi \right. \\ &\quad \left. - g(Y, \phi(\nabla_Xh)Z)\xi + g(X, \phi hZ)\phi hY - g(Y, \phi hZ)\phi hX \right) \\ &\quad - \frac{(\alpha-1)^2}{\alpha} \left(\eta(X)g(hZ, Y)\xi - \eta(Y)g(hZ, X)\xi \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2n+4} \left\{ \frac{\alpha-1}{\alpha} \left(\eta(X)\eta(Z)g(Y, Q\xi)\xi - \eta(Y)\eta(Z)g(X, Q\xi)\xi \right. \right. \\
& + g(Y, Z)g(X, Q\xi)\xi - g(X, Z)g(Y, Q\xi)\xi + g(\phi Y, Z)g(\phi X, Q\xi)\xi \\
& \quad \left. - g(\phi X, Z)g(\phi Y, Q\xi)\xi - 2g(\phi X, Y)g(\phi Z, Q\xi)\xi \right) \\
& \quad + 2n(\alpha-1) \left(\eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \right) \\
& \quad - (g(X, Z) - \eta(X)\eta(Z)) \left(2n \frac{(\alpha-1)^2}{\alpha} \eta(Y)\xi \right. \\
& \quad \left. + \frac{\alpha-1}{\alpha} (-Y + \eta(Y)\xi - 2hY + h^2Y + R(Y, \xi)\xi) \right) \\
& \quad + \frac{\alpha-1}{\alpha} \left(-g(Y, Z) + \eta(Y)\eta(Z) - 2g(hY, Z) + g(hY, hZ) \right. \\
& \quad \left. + g(R(Y, \xi)\xi, Z) \right) X + (g(Y, Z) - \eta(Y)\eta(Z)) \\
& \quad \times \left(2n \frac{(\alpha-1)^2}{\alpha} \eta(X)\xi + \frac{\alpha-1}{\alpha} (-X + \eta(X)\xi - 2hX + h^2X + R(X, \xi)\xi) \right) \\
& - \frac{\alpha-1}{\alpha} \left(-g(X, Z) + \eta(X)\eta(Z) - 2g(hX, Z) + g(hX, hZ) + g(R(X, \xi)\xi, Z) \right) Y \\
& \quad - \frac{\alpha-1}{\alpha} g(\phi X, Z) \left(-\phi Y - 2h\phi Y + h^2\phi Y + R(\phi Y, \xi)\xi \right) \\
& + \frac{\alpha-1}{\alpha} \left(-g(\phi Y, Z) - 2g(h\phi Y, Z) + g(h\phi Y, hZ) + g(R(\phi Y, \xi)\xi, Z) \right) \phi X \\
& \quad + \frac{\alpha-1}{\alpha} g(\phi Y, Z) \left(-\phi X - 2h\phi X + h^2\phi X + R(\phi X, \xi)\xi \right) \\
& - \frac{\alpha-1}{\alpha} \left(-g(\phi X, Z) - 2g(h\phi X, Z) + g(h\phi X, hZ) + g(R(\phi X, \xi)\xi, Z) \right) \phi Y \\
& - \frac{\alpha-1}{\alpha} \left(-2g(\phi X, Y) - 2g(\phi hX, hY) + g(R(\phi X, \xi)\xi, Y) - g(R(\phi Y, \xi)\xi, X) \right) \phi Z \\
& \quad - 2 \frac{\alpha-1}{\alpha} g(\phi X, Y) \left(-\phi Z - 2h\phi Z + h^2\phi Z + R(\phi Z, \xi)\xi \right) \\
& + \frac{\alpha-1}{\alpha} \left(-g(X, Z) + \eta(X)\eta(Z) + 2g(hX, Z) + g(hX, hZ) \right. \\
& \quad \left. + g(R(X, \xi)\xi, Z) \right) \eta(Y)\xi - \frac{\alpha-1}{\alpha} \left(-g(Y, Z) + \eta(Y)\eta(Z) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. -2g(hY, Z) + g(hY, hZ) + g(R(Y, \xi)\xi, Z) \right) \eta(X)\xi \Big\} \\
& - \frac{1}{(2n+4)(2n+2)} \frac{\alpha-1}{\alpha} \operatorname{Tr} h^2 \Big(g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\
& + 2g(\phi X, Y)\phi Z + g(X, Z)Y - g(Y, Z)X - \eta(Y)g(X, Z)\xi \\
& + \eta(X)g(Y, Z)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \Big)
\end{aligned}$$

Now we shall introduce in M an E -tensor by

$$\begin{aligned}
(2.8) \quad E(X, Y)Z &= \tilde{B}(X, Y)Z - \eta(X)\tilde{B}(\xi, Y)Z - \eta(Y)\tilde{B}(X, \xi)Z \\
&- \eta(Z)\tilde{B}(X, Y)\xi - \eta(\tilde{B}(X, Y)Z)\xi - \tilde{B}(\phi X, \phi Y)Z + \eta(Z)\tilde{B}(\phi X, \phi Y)\xi \\
&+ \eta(\tilde{B}(\phi X, \phi Y)Z)\xi - \phi\tilde{B}(X, \phi Y)Z - \phi\tilde{B}(\phi X, Y)Z + \eta(X)\phi\tilde{B}(\xi, \phi Y)Z \\
&+ \eta(Y)\phi\tilde{B}(\phi X, \xi)Z + \eta(Z)\phi\tilde{B}(X, \phi Y)\xi + \eta(Z)\phi\tilde{B}(\phi X, Y)\xi \\
&+ \eta(X)\eta(\tilde{B}(\xi, Y)Z)\xi + \eta(Y)\eta(\tilde{B}(X, \xi)Z)\xi.
\end{aligned}$$

In particular if M is Sasakian, $E = 0$ from (2.2).

Theorem 2.1. *The E -tensor is invariant with respect to the D -homothetic deformation $M(\phi, \xi, \eta, g) \rightarrow M(\phi^*, \xi^*, \eta^*, g^*)$ on a contact metric manifold M .*

PROOF. Using (1.4), (1.5), (1.6) and (2.7), we find

$$\begin{aligned}
(2.9) \quad -\eta^*(X)\tilde{B}^*(\xi^*, Y)Z &= -\eta(X)\tilde{B}^*(\xi, Y)Z = -\eta(X)\tilde{B}(\xi, Y)Z \\
&+ (\alpha-1) \Big(-\eta(X)\eta(Z)Y + \eta(X)\eta(Z)hY + \eta(X)(\nabla_Y \phi)Z \Big) \\
&+ \frac{\alpha-1}{\alpha} \Big(\eta(X)\eta(Y)\eta(Z)\xi + \eta(X)g(Y, R(Z, \xi)\xi)\xi \Big) \\
&+ \frac{(\alpha-1)^2}{\alpha} \eta(X)g(Y, Z)\xi + \frac{(\alpha-1)(\alpha-2)}{\alpha} \eta(X)g(Y, hZ)\xi \\
&- \frac{1}{2n+4} \frac{\alpha-1}{\alpha} \Big(g(Y, Z) - \eta(Y)\eta(Z) \Big) \Big(g(Q\xi, \xi) - 2n \Big) \eta(X)\xi
\end{aligned}$$

$$\begin{aligned}
(2.10) \quad -\eta^*(Y)\tilde{B}^*(X, \xi^*)Z &= -\eta(Y)\tilde{B}(X, \xi)Z \\
&+ (\alpha-1) \Big(\eta(Y)\eta(Z)X - \eta(Y)\eta(Z)hX - \eta(Y)(\nabla_X \phi)Z \Big) \\
&- \frac{\alpha-1}{\alpha} \Big(\eta(X)\eta(Y)\eta(Z)\xi + \eta(Y)g(X, R(Z, \xi)\xi)\xi \Big)
\end{aligned}$$

$$\begin{aligned}
& -\frac{(\alpha-1)^2}{\alpha}\eta(Y)g(Z, X)\xi - \frac{(\alpha-1)(\alpha-2)}{\alpha}\eta(Y)g(hZ, X)\xi \\
& + \frac{1}{2n+4}\frac{\alpha-1}{\alpha}\left(g(X, Z) - \eta(X)\eta(Z)\right)\left(g(Q\xi, \xi) - 2n\right)\eta(Y)\xi \\
(2.11) \quad & -\eta^*(Z)\tilde{B}^*(X, Y)\xi^* = -\eta(Z)\tilde{B}(X, Y)\xi \\
& + (\alpha-1)\left(-\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X - \eta(Y)\eta(Z)hX \right. \\
& \left. + \eta(X)\eta(Z)hY - \eta(Z)(\nabla_X\phi)Y + \eta(Z)(\nabla_Y\phi)X\right)
\end{aligned}$$

$$\begin{aligned}
(2.12) \quad & -\eta^*(\tilde{B}^*(X, Y)Z)\xi^* = -\eta(\tilde{B}^*(X, Y)Z)\xi \\
& = -\eta(\tilde{B}(X, Y)Z)\xi - \frac{\alpha-1}{\alpha}\left(\eta(X)g(Z, hY)\xi - \eta(Y)g(Z, hX)\xi \right. \\
& \quad -g(X, (\nabla_Y\phi)hZ)\xi + g(Y, (\nabla_X\phi)hZ)\xi - g(X, \phi(\nabla_Yh)Z)\xi \\
& \quad \left. + g(Y, \phi(\nabla_Xh)Z)\xi\right) + \frac{1}{2n+4}\left\{\frac{\alpha-1}{\alpha}\left(-\eta(X)\eta(Z)g(Y, Q\xi)\xi \right. \right. \\
& \quad \left. + \eta(Y)\eta(Z)g(X, Q\xi)\xi - g(Y, Z)g(X, Q\xi)\xi + g(X, Z)g(Y, Q\xi)\xi \right. \\
& \quad \left. -g(\phi Y, Z)g(\phi X, Q\xi)\xi + g(\phi X, Z)g(\phi Y, Q\xi)\xi + 2g(\phi X, Y)g(\phi Z, Q\xi)\xi\right) \\
& \quad -2n(\alpha-1)\left(g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\right) \\
& \quad + 2n\frac{(\alpha-1)^2}{\alpha}\left(g(X, Z) - \eta(X)\eta(Z)\right)\eta(Y)\xi \\
& \quad + \frac{\alpha-1}{\alpha}\left(g(Y, Z) - \eta(Y)\eta(Z) + 2g(hY, Z) - g(hY, hZ) \right. \\
& \quad \left. -g(R(Y, \xi)\xi, Z)\right)\eta(X)\xi - 2n\frac{(\alpha-1)^2}{\alpha}\left(g(Y, Z) - \eta(Y)\eta(Z)\right)\eta(X)\xi \\
& \quad - \frac{\alpha-1}{\alpha}\left(g(X, Z) - \eta(X)\eta(Z) + 2g(hX, Z) - g(hX, hZ) \right. \\
& \quad \left. -g(R(X, \xi)\xi, Z)\right)\eta(Y)\xi + \frac{\alpha-1}{\alpha}\left(g(X, Z) - \eta(X)\eta(Z) - 2g(hX, Z) \right. \\
& \quad \left. -g(hX, hZ) - g(R(X, \xi)\xi, Z)\right)\eta(Y)\xi - \frac{\alpha-1}{\alpha}\left(g(Y, Z) - \eta(Y)\eta(Z) \right.
\end{aligned}$$

$$+2g(hY, Z) - g(hY, hZ) - g(R(Y, \xi)\xi, Z)\eta(X)\xi\Big\}.$$

From (2.7), (2.11) and (2.12), we get

$$\begin{aligned}
(2.13) \quad & -\tilde{B}^*(\phi^*X, \phi^*Y)Z = -\tilde{B}(\phi X, \phi Y)Z \\
& +(\alpha-1)\Big(-\eta(Z)(\nabla_{\phi X}\phi)\phi Y + \eta(Z)(\nabla_{\phi Y}\phi)\phi X\Big) \\
& +\frac{\alpha-1}{\alpha}\Big(g(\phi X, (\nabla_{\phi Y}\phi)hZ)\xi - g(\phi Y, (\nabla_{\phi X}\phi)hZ)\xi \\
& +g(\phi X, \phi(\nabla_{\phi Y}h)Z)\xi - g(\phi Y, \phi(\nabla_{\phi X}h)Z)\xi + g(X, hZ)hY - g(Y, hZ)hX\Big) \\
& +\frac{1}{2n+4}\Big\{\frac{\alpha-1}{\alpha}\Big(-g(\phi Y, Z)g(\phi X, Q\xi)\xi + g(\phi X, Z)g(\phi Y, Q\xi)\xi \\
& +2g(\phi X, Y)g(\phi Z, Q\xi)\xi - g(Y, Z)g(X, Q\xi)\xi + g(X, Z)g(Y, Q\xi)\xi \\
& -g(X, Z)\eta(Y)g(Q\xi, \xi)\xi + g(Y, Z)\eta(X)g(Q\xi, \xi)\xi \\
& -g(Y, Q\xi)\eta(X)\eta(Z)\xi + g(X, Q\xi)\eta(Y)\eta(Z)\xi\Big) \\
& +\frac{\alpha-1}{\alpha}g(\phi X, Z)\Big(-\phi Y - 2h\phi Y + h^2\phi Y + R(\phi Y, \xi)\xi\Big) \\
& -\frac{\alpha-1}{\alpha}\Big(-g(\phi Y, Z) - 2g(h\phi Y, Z) + g(h\phi Y, hZ) + g(R(\phi Y, \xi)\xi, Z)\Big)\phi X \\
& -\frac{\alpha-1}{\alpha}g(\phi Y, Z)\Big(-\phi X - 2h\phi X + h^2\phi X + R(\phi X, \xi)\xi\Big) \\
& +\frac{\alpha-1}{\alpha}\Big(-g(\phi X, Z) - 2g(h\phi X, Z) + g(h\phi X, hZ) + g(R(\phi X, \xi)\xi, Z)\Big)\phi Y \\
& +\frac{\alpha-1}{\alpha}\Big(g(X, Z) - \eta(X)\eta(Z)\Big)\Big(-Y + \eta(Y)\xi - hY + h^2Y + R(Y, \xi)\xi\Big) \\
& -\frac{\alpha-1}{\alpha}\Big(-g(Y, Z)X + \eta(Y)\eta(Z)X - 2g(hY, Z)X + g(hY, hZ)X \\
& +g(R(Y, \xi)\xi, Z)X + g(Y, Z)\eta(X)\xi - \eta(X)\eta(Y)\eta(Z)\xi \\
& +2g(hY, Z)\eta(X)\xi - g(hY, hZ)\eta(X)\xi - g(R(Y, \xi)\xi, Z)\eta(X)\xi\Big) \\
& -\frac{\alpha-1}{\alpha}\Big(g(Y, Z) - \eta(Y)\eta(Z)\Big)\Big(-X + \eta(X)\xi - 2hX + h^2X + R(Y, \xi)\xi\Big) \\
& +\frac{\alpha-1}{\alpha}\Big(-g(X, Z)Y + \eta(X)\eta(Z)Y - 2g(hX, Z)Y + g(hX, hZ)Y
\end{aligned}$$

$$\begin{aligned}
& +g(R(X, \xi)\xi, Z)Y + g(X, Z)\eta(Y)\xi - \eta(X)\eta(Y)\eta(Z)\xi \\
& +2g(hX, Z)\eta(Y)\xi - g(hX, hZ)\eta(Y)\xi - g(R(X, \xi)\xi, Z)\eta(Y)\xi \\
& +\frac{\alpha-1}{\alpha} \left(2g(X, \phi Y) - 2g(hX, h\phi Y) - g(R(X, \xi)\xi, \phi Y) + g(R(Y, \xi)\xi, \phi X) \right) \phi Z \\
& +2\frac{\alpha-1}{\alpha} g(\phi X, Y) \left(-\phi Z - 2h\phi Z + h^2\phi Z + R(\phi Z, \xi)\xi \right) \Big\} \\
& +\frac{1}{(2n+4)(2n+2)} \frac{\alpha-1}{\alpha} \text{Tr } h^2 \left(g(X, Z)Y - g(Y, Z)X \right. \\
& \quad \left. +g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X \right. \\
& \quad \left. -\eta(X)\eta(Z)Y + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X - 2g(X, \phi Y)\phi Z \right)
\end{aligned}$$

$$\begin{aligned}
(2.14) \quad & \eta^*(Z)\tilde{B}^*(\phi^*X, \phi^*Y)\xi^* = \eta(Z)\tilde{B}(\phi X, \phi Y)\xi \\
& +(\alpha-1) \left(\eta(Z)(\nabla_{\phi X}\phi)\phi Y - \eta(Z)(\nabla_{\phi Y}\phi)\phi X \right)
\end{aligned}$$

$$\begin{aligned}
(2.15) \quad & \eta^*(\tilde{B}^*(\phi^*X, \phi^*Y)Z)\xi^* = \eta(\tilde{B}(\phi X, \phi Y)Z)\xi \\
& -\frac{\alpha-1}{\alpha} \left(g(\phi X, (\nabla_{\phi Y}\phi)hZ)\xi - g(\phi Y, (\nabla_{\phi X}\phi)hZ)\xi \right. \\
& \quad \left. +g(\phi X, \phi(\nabla_{\phi Y}h)Z)\xi - g(\phi Y, \phi(\nabla_{\phi X}h)Z)\xi \right) \\
& +\frac{1}{2n+4} \frac{\alpha-1}{\alpha} \left(g(\phi Y, Z)g(\phi X, Q\xi)\xi - g(\phi X, Z)g(\phi Y, Q\xi)\xi \right. \\
& \quad -2g(\phi X, Y)g(\phi Z, Q\xi)\xi + g(Y, Z)g(X, Q\xi)\xi - g(X, Z)g(Y, Q\xi)\xi \\
& \quad +g(X, Z)\eta(Y)g(Q\xi, \xi)\xi - g(Y, Z)\eta(X)g(Q\xi, \xi)\xi \\
& \quad \left. +g(Y, Q\xi)\eta(X)\eta(Z)\xi - g(X, Q\xi)\eta(Y)\eta(Z)\xi \right).
\end{aligned}$$

Using (1.4), (1.5) and (2.12), we have

$$\begin{aligned}
(2.16) \quad & \eta^*(X)\eta^*(\tilde{B}^*(\xi^*, Y)Z)\xi^* + \eta^*(Y)\eta^*(\tilde{B}^*(X, \xi^*)Z)\xi^* \\
& = \frac{\alpha-1}{\alpha} \left(2\eta(X)g(Y, hZ)\xi - 2\eta(Y)g(X, hZ)\xi + \eta(X)g(Y, Z)\xi \right. \\
& \quad \left. -\eta(Y)g(X, Z)\xi - \eta(X)g(Y, R(Z, \xi)\xi)\xi + \eta(Y)g(X, R(Z, \xi)\xi)\xi \right)
\end{aligned}$$

$$+\frac{1}{2n+4}\frac{\alpha-1}{\alpha}\left(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right)(g(Q\xi,\xi)-2n)\xi.$$

Thus using (2.7), (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) and (2.16) we get

$$\begin{aligned}
(2.17) \quad & \tilde{B}^*(X,Y)Z - \eta^*(X)\tilde{B}^*(\xi^*,Y)Z - \eta^*(Y)\tilde{B}^*(X,\xi^*)Z \\
& - \eta^*(Z)\tilde{B}^*(X,Y)\xi^* - \eta^*(\tilde{B}^*(X,Y)Z)\xi^* - \tilde{B}^*(\phi^*X,\phi^*Y)Z \\
& + \eta^*(Z)\tilde{B}^*(\phi^*X,\phi^*Y)\xi^* + \eta^*(\tilde{B}^*(\phi^*X,\phi^*Y)Z)\xi^* \\
& + \eta^*(X)\eta^*(\tilde{B}^*(\xi^*,Y)Z)\xi^* + \eta^*(Y)\eta^*(\tilde{B}^*(X,\xi^*)Z)\xi^* \\
& = \tilde{B}(X,Y)Z - \eta(X)\tilde{B}(\xi,Y)Z - \eta(Y)\tilde{B}(X,\xi)Z - \eta(Z)\tilde{B}(X,Y)\xi \\
& - \eta(\tilde{B}(X,Y)Z)\xi - \tilde{B}(\phi X,\phi Y)Z + \eta(Z)\tilde{B}(\phi X,\phi Y)\xi \\
& + \eta(\tilde{B}(\phi X,\phi Y)Z)\xi + \eta(X)\eta(\tilde{B}(\xi,Y)Z)\xi \\
& + \eta(Y)\eta(\tilde{B}(X,\xi)Z)\xi - 2(\alpha-1)\left(\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY\right) \\
& - \frac{\alpha-1}{\alpha}\left(g(Y,hZ)hX - g(X,hZ)hY + g(X,\phi hZ)\phi hY - g(Y,\phi hZ)\phi hX\right).
\end{aligned}$$

Here, substituting ϕ^*Y for Y in (2.17) and transforming (2.17) by ϕ^* , we have

$$\begin{aligned}
(2.18) \quad & -\phi^*\tilde{B}^*(X,\phi^*Y) - \phi^*\tilde{B}^*(\phi^*X,Y)Z + \eta^*(X)\phi^*\tilde{B}^*(\xi^*,\phi^*Y)Z \\
& - \eta^*(Y)\phi^*\tilde{B}^*(\phi^*X,\xi^*)Z - \eta^*(Z)\phi^*\tilde{B}^*(X,\phi^*Y)\xi^* \\
& - \eta^*(Z)\phi^*\tilde{B}^*(\phi^*X,Y)\xi^* - \eta^*(Y)\eta^*(Z)\phi^*\tilde{B}^*(\phi^*X,\xi^*)\xi^* \\
& = -\phi\tilde{B}(X,\phi Y) - \phi\tilde{B}(\phi X,Y)Z + \eta(X)\phi\tilde{B}(\xi,\phi Y)Z \\
& - \eta(Y)\phi\tilde{B}(\phi X,\xi)Z - \eta(Z)\phi\tilde{B}(X,\phi Y)\xi - \eta(Z)\phi\tilde{B}(\phi X,Y)\xi \\
& - \eta(Y)\eta(Z)\phi\tilde{B}(\phi X,\xi)\xi - 2(\alpha-1)\eta(X)\eta(Z)hY \\
& - \frac{\alpha-1}{\alpha}\left(g(X,\phi hZ)h\phi Y - g(Y,hZ)hX + g(Y,\phi hZ)\phi hX + g(X,hZ)hY\right).
\end{aligned}$$

On the other hand we find by virtue of (1.4)

$$\begin{aligned}
(2.19) \quad & -\eta^*(Y)\eta^*(Z)\phi^*\tilde{B}^*(\phi^*X,\xi^*)\xi^* \\
& = -\eta(Y)\eta(Z)\phi\tilde{B}(\phi X,\xi)\xi - 2(\alpha-1)\eta(Y)\eta(Z)hX.
\end{aligned}$$

Substituting (2.19) into (2.18) and combining (2.17) and (2.18), we obtain our result.

3. Contact metric manifolds with vanishing E -tensor

We define

$$(3.1) \quad s^\# = \sum_{i,j=1}^{2n+1} g(R(E_i, E_j)\phi E_j, \phi E_i),$$

where $\{E_i\}$ is an orthonormal frame.

Lemma 3.1. ([4]). *For any $(2n+1)$ -dimensional contact metric manifold M we have*

$$s^\# - S + 4n^2 = \text{Tr } h^2 + \frac{1}{2} \{\|\nabla \phi\|^2 - 4n\} \geq 0.$$

Moreover M is Sasakian if and only if $\|\nabla \phi\|^2 - 4n = 0$ or equivalently

$$s^\# - S + 4n^2 = 0.$$

Proposition 3.1. *Let M be a $(2n+1)$ -dimensional contact metric manifold with vanishing E -tensor. Then we have*

$$2 \text{Tr } h^2 = \|\nabla \phi\|^2 - 4n.$$

PROOF. Since the E -tensor of M vanishes, we have

$$(3.2) \quad \begin{aligned} g(\tilde{B}(X, Y)Z, W) &= \eta(X)g(\tilde{B}(\xi, Y)Z, W) \\ &+ \eta(Y)g(\tilde{B}(X, \xi)Z, W) + \eta(Z)g(\tilde{B}(X, Y)\xi, W) + \eta(W)\eta(\tilde{B}(X, Y)Z) \\ &+ g(\tilde{B}(\phi X, \phi Y)Z, W) - \eta(Z)g(\tilde{B}(\phi X, \phi Y)\xi, W) - \eta(W)\eta(\tilde{B}(\phi X, \phi Y)Z) \\ &- \eta(X)\eta(W)\eta(\tilde{B}(\xi, Y)Z) - \eta(Y)\eta(W)\eta(\tilde{B}(X, \xi)Z) + g(\phi \tilde{B}(X, \phi Y)Z, W) \\ &+ g(\phi \tilde{B}(\phi X, Y)Z, W) - \eta(X)g(\phi \tilde{B}(\xi, \phi Y)Z, W) - \eta(Y)g(\phi \tilde{B}(\phi X, \xi)Z, W) \\ &- \eta(Z)g(\phi \tilde{B}(X, \phi Y)\xi, W) - \eta(Z)g(\phi \tilde{B}(\phi X, Y)\xi, W) \end{aligned}$$

Taking $X = E_i$, $Y = E_j$, $Z = \phi E_j$, $W = \phi E_i$, ($\{E_i\}$ is a ϕ -basis) into each member of (3.2) and summing over i and j , we have

$$\begin{aligned} \sum_{i,j=1}^{2n+1} g(\tilde{B}(E_i, E_j)\phi E_j, \phi E_i) &= s^\# - S - \frac{2(n+1)}{n+2} \text{Tr } h^2 + 4n^2 \\ \sum_{i,j=1}^{2n+1} g(\tilde{B}(\phi E_i, \phi E_j)\phi E_j, \phi E_i) &= S - 2g(Q\xi, \xi) \\ &+ \frac{2(n+1)}{n+2} (g(Q\xi, \xi) - S) + \frac{3n(k+2n)}{n+2} + \frac{n(2n-1)(k-4)}{n+2} \end{aligned}$$

$$\begin{aligned}
\sum_{i,j=1}^{2n+1} g(\phi \tilde{B}(E_i, \phi E_j) \phi E_j, \phi E_i) &= S - 2g(Q\xi, \xi) \\
&\quad + \frac{2(n+1)}{n+2} (g(Q\xi, \xi) - S) + \frac{3n(k+2n)}{n+2} + \frac{n(2n-1)(k-4)}{n+2} \\
\sum_{i,j=1}^{2n+1} g(\phi \tilde{B}(\phi E_i, E_j) \phi E_j, \phi E_i) &= -s^\# - \frac{2(n+1)}{n+2} (g(Q\xi, \xi) - S) \\
&\quad - \frac{n(2n+1)}{n+2} (k+2n) - \frac{n(k-4)}{n+2} \\
\sum_{i,j=1}^{2n+1} \eta(E_i) g(\tilde{B}(\xi, E_j) \phi E_j, \phi E_i) &= 0 \\
\sum_{i,j=1}^{2n+1} \eta(E_j) g(\tilde{B}(E_i, \xi) \phi E_j, \phi E_i) &= 0 \\
\sum_{i,j=1}^{2n+1} \eta(E_i) g(\phi \tilde{B}(\xi, \phi E_j) \phi E_j, \phi E_i) &= 0 \\
\sum_{i,j=1}^{2n+1} \eta(E_j) g(\phi \tilde{B}(\phi E_j, \xi) \phi E_j, \phi E_i) &= 0,
\end{aligned}$$

where we have used

$$\begin{aligned}
\sum_{i,j=1}^{2n+1} g(R(\phi E_i, E_j) \phi E_j, E_i) &= - \sum_{i,j=1}^{2n+1} g(\phi E_j, R(\phi E_i, E_j) E_i) \\
&= \sum_{i,j=1}^{2n+1} \left(g(\phi E_j, R(E_j, E_i) \phi E_i) + g(\phi E_j, R(E_i, \phi E_i) E_j) \right) \\
&= \sum_{i,j=1}^{2n+1} \left(g(\phi E_j, R(E_j, E_i) \phi E_i) - g(\phi E_j, R(\phi E_i, E_j) E_i) \right. \\
&\quad \left. - g(\phi E_j, R(E_j, E_i) \phi E_i) \right) \\
&= \sum_{i,j=1}^{2n+1} \left(g(\phi E_j, R(E_j, E_i) \phi E_i) + g(\phi E_j, R(E_j, \phi E_i) E_i) \right. \\
&\quad \left. - g(\phi E_j, R(E_j, E_i) \phi E_i) \right) = -s^\#
\end{aligned}$$

Therefore, using each of the above equations, (1.2) and $k = \frac{S+2n}{2n+2}$, we get

$$s^\# - S + 4n^2 = 2 \operatorname{Tr} h^2.$$

From Lemma 3.1, we have our result.

By Proposition 3.1, we immediately get the following

Theorem 3.1. *Let M be a K -contact Riemannian manifold. Then M is a Sasakian manifold if and only if the E -tensor vanishes.*

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