

Lie ideals and commuting mappings in prime rings

By RAM AWTAR (Pilani)

Abstract. J. VUKMAN [6, Theorem 1] proved that if R is a prime ring of characteristic different from two and if d is a derivation of R such that $[[d(x), x], x] = 0$ for all x in R then either $d = 0$ or R is commutative. This result extends one due to E. POSNER [5, Theorem 2]. In this paper our object is to generalize the above mentioned result of J. VUKMAN to Lie ideals and we prove the following:

Theorem. Let R be a prime ring of characteristic different from two, and let d be a derivation of R . Let U be a Lie ideal of R such that $[[d(u), u], u] = 0$ for all $u \in U$. Then either $d = 0$ or $U \subset Z$, where Z is the center of R .

1. Introduction

A theorem of J. VUKMAN [6] states that if R is a prime ring of characteristic different from 2, and if d is a derivation of R such that $(d(x)x - xd(x))x - x(d(x)x - xd(x)) = 0$ for all x in R then either $d = 0$ or R is commutative. In this paper we extend this result to Lie ideals.

Throughout the paper we assume that R is an associative ring of characteristic not 2. The center of R is denoted by Z . We also assume that d is a derivation of R , i.e., an additive mapping of R into itself such that $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. For $x, y \in R$, let

$$[x, y] = xy - yx \quad \text{and} \quad f(x) = [x, d(x)]$$

An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$.

2. The main theorem

In this section we prove the main theorem of this paper which generalizes a result of J. VUKMAN [6, theorem 1] to Lie ideals. We begin with the main theorem.

Theorem 1. *Let R be a prime ring, $\text{char} \neq 2$, and let U be a Lie ideal of R . If d is a derivation of R such that $[[d(u), u], u] = 0$ for all $u \in U$, then either $d = 0$ or $U \subset Z$.*

PROOF. By hypothesis,

$$(1) \quad [[d(u), u], u] = 0 \quad \text{for all } u \in U.$$

In (1), replace u by $u + v$ where $v \in U$ and use (1) to get

$$\begin{aligned} & [[d(u), u], v] + [[d(u), v], u] + [[d(v), u], u] + [[d(v), v], u] + \\ & + [[d(v), u], v] + [[d(u), v], v] = 0. \end{aligned}$$

Replace u by $-u$ then

$$\begin{aligned} & [[d(u), u], v] + [[d(u), v], u] + [[d(v), u], u] - [[d(v), v], u] - \\ & - [d(v), u], v] - [[d(u), v], v] = 0. \end{aligned}$$

Adding the last two equations and dividing by 2, we have

$$(2) \quad [[d(u), u], v] + [[d(u), v], u] + [[d(v), u], u] = 0$$

for all $u, v \in U$.

Suppose now that $v, w \in U$ are such that vw is also in U . By replacing v by vw in (2), where $w \in U$, and after expanding we get

$$\begin{aligned} & v[[d(u), u], w] + [[d(u), u], v]w + v[[d(u), w], u] + [v, u][d(u), w] + \\ & + [d(u), v][w, u] + [[d(u), v], u]w + d(v)[[w, u], u] + [d(v), u][w, u] + \\ & + [d(v), u][w, u] + [[d(v), u], u]w + v[[d(w), u], u] + [v, u][d(w), u] + \\ & + [v, u][d(w), u] + [[v, u], u]d(w) = 0. \end{aligned}$$

In view of (2), the last equation reduces to

$$\begin{aligned} & [v, u][d(u), w] + [d(u), v][w, u] + d(v)[[w, u], u] + 2[d(v), u][w, u] + \\ & + 2[v, u][d(w), u] + [[v, u], u]d(w) = 0. \end{aligned}$$

For any $r \in R$, the element $w = vr - rv$ satisfies the criterion $vw \in U$, hence by the above equation, we get

$$(3) \quad \begin{aligned} & [v, u][d(u), [v, r]] + [d(u), v][[v, r], u] + d(v)[[[v, r], u], u] + \\ & + 2[d(v), u][[v, r], u] + 2[v, u][d[v, r], u] + [[v, u], u]d[v, r] = 0 \end{aligned}$$

for all $u, v \in U, r \in R$.

Let $v = u$ in (3). Then

$$3[u, d(u)][u, [u, r]] + d(u)[u, [u, [u, r]]] = 0.$$

For $u \in U$, let $I(r) = [u, r]$ for all $r \in R$. Then I is an inner derivation of R .

Thus from the above equation, we get

$$(4) \quad 3f(u)I^2(r) + d(u)I^3(r) = 0 \quad \text{for all } r \in R, u \in U.$$

In (4), let $r = ur$. Then, as $I(u) = 0$,

$$3f(u)uI^2(r) + d(u)uI^3(r) = 0.$$

From (1) $uf(u) = f(u)u$. So by (4) and the last equation, we get

$$(5) \quad f(u)I^3(r) = 0 \quad \text{for all } r \in R, u \in U.$$

Similarly, after replacing v by $wv = [v, r]v$ in (2), and then putting $v = u$, we obtain

$$(6) \quad 3I^2(r)f(u) + I^3(r)d(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Replace r by $rd(u)$ in (4) and use (4) to get

$$3f(u) \{2I(r)I(d(u)) + rI^2(d(u))\} + d(u) \{3I^2(r)I(d(u)) + 3I(r)I^2(d(u)) + rI^3(d(u))\} = 0.$$

In view of (1), $I^2(d(u)) = 0$, and $I(d(u)) = [u, d(u)] = f(u)$. Thus, we get

$$(7) \quad 6f(u)I(r)f(u) + 3d(u)I^2(r)f(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Similarly, after replacing r by $d(u)r$ in (4), we have

$$(8) \quad 6f(u)f(u)I(r) + 3f(u)d(u)I^2(r) = 0 \quad \text{for all } r \in R, u \in U.$$

In (6) replace r by $rd(u)$ and use (6). Then

$$(9) \quad 6I(r)f(u)f(u) + 3I^2(r)f(u)d(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Similarly, after replacing r by $d(u)r$ in (6), we get

$$(10) \quad 6f(u)I(r)f(u) + 3f(u)I^2(r)d(u) = 0 \quad \text{for all } r \in R, u \in U.$$

In (4), let $r = rs$ where $s \in R$ and use (4) to get

$$(11) \quad 3f(u)\{2I(r)I(s) + rI^2(s)\} + d(u)\{3I^2(r)I(s) + 3I(r)I^2(s) + rI^3(s)\} = 0 \quad \text{for all } r, s \in R, u \in U.$$

Replace r by $f(u)$ in (11). Then, as $I(f(u)) = 0$ from (1) and $f(u)I^3(s) = 0$ from (5), we get

$$(12) \quad 3f(u)f(u)I^2(s) = 0 \quad \text{for all } s \in R, u \in U.$$

Write $s = rs$, $r \in R$ in (12) and use (12) to obtain

$$(13) \quad 3f(u)f(u)\{2I(r)I(s) + rI^2(s)\} = 0 \quad \text{for all } r, s \in R, u \in U.$$

From (11) and (13), we get

$$(14) \quad f(u)d(u)\{3I^2(r)I(s) + 3I(r)I^2(s) + rI^3(s)\} = 0 \\ \text{for all } r, s \in R, u \in U.$$

Replacing r by ur in (14) and using (14), as $uf(u) = f(u)u$ from (1), we get

$$f(u)f(u)\{3I^2(r)I(s) + 3I(r)I^2(s) + rI^3(s)\} = 0.$$

In view of (12), the last equation reduces to

$$f(u)f(u)\{3I(r)I^2(s) + rI^3(s)\} = 0$$

i.e.,

$$2f(u)f(u)\{3I(r)I^2(s) + rI^3(s)\} = 0.$$

In (13), putting $s = I(s)$ and then subtracting, from the last equation, we have $f(u)f(u)RI^3(s) = 0$. Since R is prime, if $f(u)f(u) \neq 0$ for some u in U , then $I^3(s) = 0$ for all $s \in R$. Thus, from (6), $3I^2(r)f(u) = 0$ and so, from (9), $6I(r)f(u)f(u) = 0$, i.e., $3I(r)f(u)f(u) = 0$ for all $r \in R$. Suppose R is of characteristic different from 3. Then $I(r)f(u)f(u) = 0$ and so $0 = \{I(r)s + rI(s)\}f(u)f(u) = I(r)Rf(u)f(u) = [u, R]Rf(u)f(u)$. Since $f(u)f(u) \neq 0$ and R is prime, then $u \in Z$ and so $f(u)f(u) = 0$. Thus $f(u)f(u) = 0$ for all $u \in U$.

Suppose now that R is of characteristic 3. Then from (4), $d(u)I^3(r) = 0$. Thus

$$(15) \quad 0 = d(u)[u, [u, [u, r]]] = d(u)[u^3, r] \quad \text{for all } r \in R, u \in U.$$

In (15), replace u by $u + v$ where $v \in U$ and use (15). Then

$$\{d(u) + d(v)\}[u^2v + uvu + vu^2 + uv^2 + vuv + v^2u, r] + \\ + d(u)[v^3, r] + d(v)[u^3, r] = 0.$$

Replace u by $-u$, then

$$\{-d(u) + d(v)\}[u^2v + uvu + vu^2 - uv^2 - vuv - v^2u, r] - \\ - d(u)[v^3, r] - d(v)[u^3, r] = 0.$$

Adding the last two equations and dividing by 2, we have

$$(16) \quad d(u)[uv^2 + vuv + v^2u, r] + d(v)[u^2v + uvu + vu^2, r] = 0$$

for all $r \in R$ and $u, v \in U$.

Replacing r by ru in (16) and using (16) we get

$$d(u)r[uv^2 + vuv + v^2u, u] + d(v)r[u^2v + uvu + vu^2, u] = 0$$

or

$$d(u)r[uv^2 + vuv + v^2u, u] + d(v)r\{(u^2v + uvu + vu^2)u - u(u^2v + uvu + vu^2)\} = 0$$

or

$$(17) \quad d(u)r[uv^2 + vuv + v^2u, u] + d(v)r[v, u^3] = 0$$

for all $r \in R$, and $u, v \in U$.

In (15) let $r = rs$, then $0 = d(u)\{[u^3, r]s + r[u^3, s]\} = d(u)R\{u^3, s\}$. So, if for some $u \in U$, $d(u) \neq 0$ then $[u^3, s] = 0$ for all $s \in R$, since R is prime. Thus from (17) $d(u)R[uv^2 + vuv + v^2u, u] = 0$, i.e., $[uv^2 + vuv + v^2u, u] = 0$ for all $v \in U$, as $d(u) \neq 0$ and R is prime. Thus

$$\begin{aligned} 0 &= I(uv^2 + vuv + v^2u) \\ &= I\{(uv - vu)v - v(uv - vu)\}, \text{ as } \text{char} R = 3, \\ &= I[I(v), v] = [I^2(v), v] \end{aligned}$$

Linearize the last equation on v to get

$$[I^2(v), w] + [I^2(w), v] = 0 \text{ for all } v, w \in U.$$

Replacing w by $I(w)$, as $I^3(w) = [u, [u, [u, w]]] = [u^3, w] = 0$, we get $[I^2(v), I(w)] = 0$ for all $v, w \in U$. Suppose that $u \notin Z$, then by Theorem 4 of [2], as $I \neq 0$ and $U \not\subset Z$, $I^2(v) \in Z$ for all $v \in U$. Now by Theorem 5 of [2], as $I \neq 0$, $U \subset Z$. Thus $u \in Z$ and so $f(u) = [u, d(u)] = 0$. Hence for all $u \in U$, $f(u) = 0$. Thus, in all cases

$$(18) \quad f(u)f(u) = 0 \text{ for all } u \in U.$$

Linearizing (3) on v , we get

$$(19) \quad \begin{aligned} &[v, u][d(u), [w, r]] + [w, u][d(u), [v, r]] + [d(u), v][[w, r], u] + \\ &+ [d(u), w][[v, r], u] + d(v)[[w, r], u] + d(w)[[v, r], u] + \\ &+ 2[d(v), u][[w, r], u] + 2[d(w), u][[v, r], u] + 2[v, u][d[w, r], u] + \\ &+ 2[w, u][d[v, r], u] + [[v, u], u]d[w, r] + [[w, u], u]d[v, r] = 0 \end{aligned}$$

for all $u, v, w \in U$.

Let $v=u=f(u)$ in (19). Then, since $f(u)f(u) = 0$ and $f(u)d(f(u)) \times f(u)=0$, we get

$$(20) \quad \begin{aligned} & [w, f(u)][d(f(u)), [f(u), r]] + 3[d(f(u)), f(u)][w, r]f(u) + \\ & + 2[d(f(u)), w] f(u)r f(u) - 2d(f(u)) f(u)[w, r] f(u) + \\ & + 4[d(w), f(u)] f(u)r f(u) + 2[w, f(u)] d[f(u), r]f(u) = 0 \\ & \text{for all } r \in R \text{ and } u, w \in U. \end{aligned}$$

Multiplying by $f(u)$ from the right, the last equation yields

$$[w, f(u)] [d(f(u)), [f(u), r]] f(u) = 0$$

i.e.,

$$\begin{aligned} & f(u)U\{f(u)r d(f(u)) f(u) - d(f(u)) f(u)r f(u)\} = 0 \\ & \text{for all } r \in R, u \in U. \end{aligned}$$

If for some $u \in U$, $f(u) \neq 0$ then $u \notin Z$ and so by Lemma 4 of [3] the last equation yields

$$f(u)r d(f(u))f(u) = d(f(u)) (f(u))r f(u) \text{ for all } r \in R.$$

By using a result of MARTINDALE (Corollary of Lemma 1.3.2 in [4]) we conclude that, as $f(u) \neq 0$, $d(f(u)) f(u) = \delta(u) f(u)$ where $\delta(u) \in C$, the extended centroid of R (See p.22 of [4] for the notion of extended centroid). Moreover, $f(u)d(f(u)) = -d(f(u))f(u) = -\delta(u) f(u)$.

Let $r = f(u)r$ in (20). Since $0 = f(u)f(u) = f(u)d(f(u)) f(u)$, $d(f(u))f(u) = \delta(u)f(u)$ and $f(u)d(f(u)) = -\delta(u)f(u)$, we get

$$\begin{aligned} & -[w, f(u)][d(f(u)), f(u)r f(u)] + 6\delta(u) f(u)w f(u)r f(u) - \\ & - 2\delta(u)f(u)w f(u)r f(u) + 2[w, f(u)] d\{f(u)[f(u), r]\}f(u) = 0 \end{aligned}$$

i.e.,

$$\begin{aligned} & f(u)w d(f(u)) f(u)r f(u) - f(u)w f(u)r f(u) d(f(u)) + \\ & + 4\delta(u)f(u)w f(u)r f(u) + 2[w, f(u)]\{d(f(u))[f(u), r] + \\ & + f(u)[d(f(u)), r] + f(u)[f(u), d(r)]\}f(u) = 0 \end{aligned}$$

i.e.,

$$\begin{aligned} & 6\delta(u)f(u)w f(u)r f(u) + 2[w, f(u)] d(f(u)) f(u)r f(u) - \\ & - 2 f(u)w f(u)[d(f(u)), r] f(u) = 0 \end{aligned}$$

i.e.

$$6\delta(u) f(u)w f(u)r f(u) - 2f(u)w d(f(u)) f(u)r f(u) - \\ - 2f(u)w f(u) d(f(u))r f(u) + 2f(u)wf(u)r d(f(u)) f(u) = 0$$

i.e.

$$8\delta(u) f(u)wf(u)r f(u) = 0 \quad \text{for all } w \in U, r \in R.$$

Now, if $\delta(u) \neq 0$ then $f(u) U f(u) R f(u) = 0$ and so $f(u) U f(u) = 0$, as $f(u) \neq 0$ and R is prime. As $f(u) \neq 0$ then $u \notin Z$, so again by Lemma 4 of [3] $f(u) = 0$, a contradiction. Thus $\delta(u) = 0$. Therefore, $d(f(u)) f(u) = f(u)d(f(u)) = 0$. Hence from (20), we get

$$(21) \quad f(u)w\{d(f(u))rf(u) + f(u)rd(f(u))\} + 2d(f(u)wf(u)r f(u) - \\ - 4f(u)d(w)f(u)r f(u) - 2f(u)w\{d(f(u))r f(u) + \\ + f(u)d(r)f(u)\} = 0 \quad \text{for } r \in R, w \in U.$$

In (2), let $u = f(u)$. Then we get

$$(22) \quad d(f(u))vf(u) + f(u)vd(f(u)) - 2f(u)d(v)f(u) = 0 \quad \text{for all } v \in U.$$

Thus from (21), we get

$$\{2d(f(u))wf(u) - 4f(u)d(w)f(u) - 2f(u)wd(f(u))\}vf(u) = 0 \\ \text{for all } v, w \in U.$$

Again, $f(u) \neq 0$ implies $u \notin Z$. Hence, by Lemma 4 of [3],

$$(23) \quad d(f(u))wf(u) - f(u)wd(f(u)) - 2f(u)d(w)f(u) = 0 \quad \text{for all } w \in U.$$

In view of (22) & (23), we get $2f(u)wd(f(u)) = 0$ for all $w \in U$. Then $f(u)Ud(f(u)) = 0$. Thus $d(f(u)) = 0$, since $f(u) \neq 0$. So from (22), $f(u)d(v)f(u) = 0$ for all $v \in U$. Hence from (21), $f(u)Uf(u)d(r)f(u) = 0$ for all $r \in R$. As $f(u) \neq 0$, we get $f(u)d(r)f(u) = 0$ for all $r \in R$. Replace r by ru , then $0 = f(u)\{d(r)u + rd(u)\}f(u) = f(u)r d(u) f(u)$, as $uf(u) = f(u)u$. Since R is prime and $f(u) \neq 0$, $d(u)f(u) = 0$. Also $0 = d[u, f(u)] = [d(u), f(u)] = f(u)d(u)$. Thus we conclude that if for some $u \in U$, $f(u) \neq 0$ then $f(u)d(u) = d(u)f(u) = 0$. Hence

$$f(u)d(u) = d(u)f(u) = 0 \quad \text{for all } u \in U$$

i.e.,

$$(24) \quad [u, d(u)]d(u) = d(u)[u, d(u)] = 0 \quad \text{for all } u \in U.$$

Linearizing (24) on u and using a similar approach as in the proof of (2) we get

$$(25) \quad [u, d(u)]d(v) + [u, d(v)]d(u) + [v, d(u)]d(u) = 0 \quad \text{for all } u, v \in U.$$

Suppose now that $v, w \in U$ are such that $vw \in U$. By replacing v by vw in (25), where $w \in U$, and using (25), after expansion we conclude

$$(26) \quad \begin{aligned} & [u, d(u)]d(v)w + [u, d(v)]wd(u) + d(v)[u, w]d(u) + \\ & + [u, v]d(w)d(u) + [v, d(u)]wd(u) + [[u, d(u)], v]d(w) = 0 \end{aligned}$$

for $u, v, w \in U$.

For any $r \in R$, the elements $w = vr - rv$ satisfies the criterion that $vw \in U$, hence by the above equation, we get

$$(27) \quad \begin{aligned} & [u, d(u)]d(v)[v, r] + [u, d(v)][v, r]d(u) + d(v)[u, [v, r]]d(u) + \\ & + [u, v]d[v, r]d(u) + [v, d(u)][v, r]d(u) + [[u, d(u)], v]d[v, r] = 0 \end{aligned}$$

for all $r \in R$ and $u, v \in U$.

Let $v = u$ in (27). Then, in view of (1) and (24), we get

$$(28) \quad 2f(u)I(r)d(u) + d(u)I^2(r)d(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Write $r = rs$ where $s \in R$ in (28). Then

$$(29) \quad \begin{aligned} & 2f(u)\{I(r)s + rI(s)\}d(u) + d(u)\{I^2(r)s + 2I(r)I(s) + \\ & + rI^2(s)\}d(u) = 0 \quad \text{for all } r, s \in R \text{ and } u \in U. \end{aligned}$$

Replace r by u in (29). Then, as $I(u) = [u, u] = 0$

$$(30) \quad 2f(u)uI(s)d(u) + d(u)uI^2(s)d(u) = 0 \quad \text{for all } s \in R, u \in U.$$

As $uf(u) = f(u)u$, so from (28) and (30), we get

$$(31) \quad f(u)I^2(r)d(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Similarly as above, from $d(u)[u, d(u)] = 0$ for all $u \in U$, we can conclude

$$(32) \quad d(u)I^2(r)f(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Let $s = u$ in (29). Then $2f(u)I(r)ud(u) + d(u)I^2(r)ud(u) = 0$. But from (28), $2f(u)I(r)d(u)u + d(u)I^2(r)d(u)u = 0$. Thus, $2f(u)I(r)f(u) + d(u)I^2(r)f(u) = 0$. Hence, in view of (32)

$$(33) \quad f(u)I(r)f(u) = 0 \quad \text{for all } r \in R, u \in U.$$

Linearize (27) on v to get

$$\begin{aligned}
 & [u, d(u)]d(v)[w, r] + [u, d(v)][w, r]d(u) + d(v)[u, [w, r]]d(u) + \\
 & + [u, v]d[w, r]d(u) + [v, d(u)][w, r]d(u) + [[u, d(u)], v]d[w, r] + \\
 (34) \quad & + [u, d(u)]d(w)[v, r] + [u, d(w)][v, r]d(u) + d(w)[u, [v, r]]d(u) + \\
 & + [u, w]d[v, r]d(u) + [w, d(u)][v, r]d(u) + [[u, d(u)], w]d[v, r] = 0 \\
 & \text{for all } r \in R \text{ and } u, v, w \in U.
 \end{aligned}$$

Write $v = u$ in (34). As $f(u)d(u) = 0$ and $[f(u), u] = 0$, we get

$$\begin{aligned}
 & 2f(u)[w, r]d(u) + d(u)[u, [w, r]]d(u) + f(u)d(w)[u, r] + \\
 & + [u, d(w)][u, r]d(u) + d(w)[u, [u, r]]d(u) + [u, w]d[u, r]d(u) + \\
 & + [w, d(u)][u, r]d(u) + [f(u), w]d[u, r] = 0.
 \end{aligned}$$

Multiplying by $f(u)$ from the right, the last equation becomes

$$f(u)d(w)[u, r]f(u) + [f(u), w]d[u, r]f(u) = 0$$

i.e.,

$$f(u)d(w)[u, r]f(u) + [f(u), w][d(u), r]f(u) + [f(u), w][u, d(r)]f(u) = 0$$

i.e.,

$$f(u)d(w)[u, r](u) + f(u)wd(u)rf(u) + [f(u), w][u, d(r)]f(u) = 0.$$

Replace r by $d(u)$ in the last equation. As $f(u)f(u) = 0$ and $d(u)f(u) = 0$, we have $0 = [f(u), w][u, d^2(u)]f(u) = [f(u), w]d[u, d(u)]f(u) = [f(u), w]d(f(u))f(u)$. Now $f(u)f(u) = 0$ and so $f(u)d(f(u))f(u) = 0$. Hence $f(u)wd(f(u))f(u) = 0$ for all $u, w \in U$. Now if for some $u \in U$, $f(u) \neq 0$ then $u \notin Z$. Thus, by Lemma 4 of [3], $d(f(u))f(u) = 0$. By multiplying $f(u)$ from the right in (34), we have

$$\begin{aligned}
 & f(u)d(v)[w, r]f(u) + [f(u), v]\{[d(w), r] + [w, d(r)]\}f(u) + \\
 & + f(u)d(w)[v, r]f(u) + [f(u), w]\{[d(v), r] + [v, d(r)]\}f(u) = 0 \\
 & \text{for all } r \in R \text{ and } v, w \in U.
 \end{aligned}$$

Replace w by $[u, d(u)] = f(u)$. As $d(f(u))f(u) = 0$, $f(u)d(f(u)) = 0$ and $f(u)f(u) = 0$, we get

$$f(u)d(v)f(u)r f(u) + [f(u), v]\{[d(f(u)), r] + [f(u), d(r)]\}f(u) = 0.$$

From (25) we have $f(u)d(v)f(u) = 0$, since $d(u)f(u) = 0$ by (24). Hence from the last equation, we get

$$[f(u), v]\{d(f(u))r f(u) + f(u)d(r)f(u)\} = 0$$

i.e.

$$f(u)U\{d(f(u))r f(u) + f(u) d(r) f(u)\} = 0 \quad \text{for all } r \in R.$$

As $f(u) \neq 0$ and so $u \notin Z$, by Lemma 4 of [3], $d(f(u)) r f(u) + f(u)d(r) \times f(u) = 0$ for all $r \in R$. In particular $d(f(u))v f(u) + f(u)d(v)f(u) = 0$ for all $v \in U$. As we have seen above, $f(u)d(v)f(u) = 0$, so we conclude that $d(f(u)) v f(u) = 0$ for all $v \in U$, i.e., $d(f(u))Uf(u) = 0$. As $f(u) \neq 0$, so by Lemma 4 of [3] $d(f(u)) = 0$.

Replace v by $f(u)$ in (34). As $d(f(u)) = 0$ and keeping in view (1) and (24), we get

$$\begin{aligned} & f(u)d(w)[f(u), r] + [u, d(w)]f(u)rd(u) + \\ (35) \quad & + d(w)[u, [f(u), r]]d(u) + [u, w][f(u), d(r)]d(u) + \\ & + [w, d(u)]f(u)rd(u) + [f(u), w][f(u), d(r)] = 0 \\ & \text{for all } r \in R, w \in U. \end{aligned}$$

Multiplying by $f(u)$ from the right in (35), as we have seen above $f(u)d(w)f(u) = 0$ for all $w \in U$, and we conclude that

$$f(u)wf(u)d(r)f(u) = 0 \quad \text{for all } w \in U, r \in R.$$

As $f(u) \neq 0$, again by Lemma 4 of [3], we have

$$(36) \quad f(u)d(r)f(u) = 0 \quad \text{for all } r \in R.$$

In view of (36), we conclude from (35) that

$$\begin{aligned} & -f(u)d(w)rf(u) + [u, d(w)]f(u)rd(u) + d(w)f(u)[u, r]d(u) + \\ (37) \quad & + [u, w]f(u)d(r)d(u) + [w, d(u)] f(u)r d(u) + f(u)w f(u) d(r) - \\ & - f(u)wd(r)f(u) = 0 \quad \text{for all } r \in R, w \in U. \end{aligned}$$

Replace r by ru in (37) and use (37). As $uf(u) = f(u)u$ and $d(u)f(u) = 0$, we get

$$\begin{aligned} & [u, d(w)]f(u)rf(u) + d(w)f(u)[u, r]f(u) + [u, w]f(u)d(r)f(u) + \\ & + [u, w] f(u)r d(u)d(u) + [w, d(u)]f(u)r f(u) + f(u)w f(u)r d(u) = 0 \\ & \text{for all } r \in R, w \in U. \end{aligned}$$

In view of (33) and (36) the last equation yields

$$(38) \quad [u, d(w)] f(u)r f(u) + [u, w] f(u)r d(u)d(u) + [w, d(u)] f(u)r f(u) + \\ + f(u)w f(u)r d(u) = 0 \quad \text{for all } r \in R, w \in U.$$

In (38), let $r = ru$, and use (38). As $uf(u) = f(u)u$, we get

$$[u, w]f(u)r u d(u)d(u) + f(u)w f(u)r f(u) - [u, w]f(u)r d(u)d(u)u = 0.$$

But from (24), $ud(u)d(u) = d(u)ud(u) = d(u)d(u)u$. Hence $f(u)w f(u)r f(u) = 0$ for all $w \in U, r \in R$. Since $f(u) \neq 0$, by Lemma 4 of [3], $f(u)r f(u) = 0$ for all $r \in R$. Since R is prime, we conclude that $f(u) = 0$. Thus $f(u) = 0$ for all $u \in U$. So, by Theorem 7 of [2] either $d = 0$ or $U \subset Z$.

3. In this section our object is to provide the affirmative answer for a question raised by J. VUKMAN in [6]. J. VUKMAN [6, Theorem 2] proved that if R is a prime ring, $\text{char} R \neq 2, 3$, and d is a derivation of R such that $[[d(x), x], x] \in Z$ for all $x \in R$, then either $d = 0$ or R is commutative. We generalize this result in case $\text{char} R = 3$ and prove the following:

Theorem 2. *Let R be a prime ring of characteristic different from 2, and let d be a derivation of R such that $[[d(x), x], x] \in Z$ for all $x \in R$. Then either $d = 0$ or R is commutative.*

PROOF. *Case I.* If $\text{char} R \neq 2, 3$, the result follows from Theorem 2 of [6].

Case II. Suppose now that $\text{char} R = 3$. Replace x by $x+y$ where $y \in R$ in the hypothesis, then by using a similar approach as in the proof of (2) we obtain

$$(39) \quad [[d(x), x], y] + [[d(x), y], x] + [[d(y), x], x] \in Z \quad \text{for all } x, y \in R.$$

Replace y by yx in (39), as $\text{char} R = 3$, and expand then

$$(40) \quad \{[[d(x), x], y] + [[d(x), y], x] + [[d(y), x], x]\}x + \\ + [[y, x], x]d(x) \in Z \quad \text{for all } x, y \in R$$

Commuting (40) with x , in view of (39), we get

$$[[y, x], x]d(x)x = x[[y, x], x]d(x) \quad \text{for all } x, y \in R.$$

Replace y by $d(x)$, then $[[d(x), x], x][d(x), x] = 0$, since $[[d(x), x], x] \in Z$. Hence $[f(x), x]Rf(x) = 0$ for all $x \in R$. If for some $x \in R$, $[f(x), x] \neq 0$ then $f(x) = 0$ and so $[f(x), x] = 0$, since R is prime. Thus $[f(x), x] = 0$ for all $x \in R$. So by Theorem 1 of [6] either $d = 0$ or R is commutative.

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RAM AWTAR
DEPARTMENT OF MATHEMATICS
BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE
PILANI (RAJ.) – 333031,
INDIA

(Received January 3, 1992; revised June 16, 1992)