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On the generalized Einstein – Yang Mills equations

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Abstract. Let $\xi = (E, p, M)$ be a vector bundle, with basis M — a real differentiable manifold of dimension n, and fiber F of dimension m. Considering the automorphisms of ξ as gauge transformations, and the set of gauge fields $\{N_i^a(x,y)$, $L_{jk}^i(x,y)$, $L_{bk}^a(x,y)$, $C_{ja}^i(x,y)$, $C_{bc}^a(x,y)$, $g_{ij}(x,y)$, $h_{ab}(x,y)\}$ given by a nonlinear connection, a gauge linear d-connection [9,11], and a pair of metric gauge tensor fields in local adapted coordinates, the author obtains the form of the generalized Einstein–Yang Mills equations for the general case and for the quasi-metric h- and v-symmetrical cases. These results generalise the ones obtained by G.S. Asanov in [2,3], in a natural manner, basically using the formalism, notations and mathematical theory of distinguished geometrical object fields introduced by R. Miron [10, 11].

Let $\{N_i^a(x,y)\}$ be the coefficients of a nonlinear connection on the vector bundle $\xi = (E, p, M)$ in local coordinates (x^i, y^a), $i = \overline{1,n}$, $a = \overline{1,m}$ [8,11].

Definition 1. A local adapted basis in $\mathfrak{X}(E)$ is the set of vector fields $\{\delta_i, \dot\partial_a\}$, $i = \overline{1,n}$, $a = \overline{1,m}$, where

$$\delta_i = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \qquad \dot\partial_a = \frac{\partial}{\partial y^a}. \tag{1}$$

Definition 2. A linear d-connection on E is a linear connection ∇ that preserves the horizontal and the vertical distributions locally generated by $\{\delta_i, i = \overline{1,n}\}$ and $\{\dot\partial_a, a = \overline{1,m}\}$ respectively; in the local adapted basis (1) its coefficients are given by

$$\{L_{jk}^i(x,y),\ L_{bk}^a(x,y),\ C_{ja}^i(x,y),\ C_{bc}^a(x,y)\}$$

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where

$$\begin{cases} \nabla_{\delta_j}\delta_i = L^k_{ij}\delta_k, \ \ \nabla_{\delta_j}\dot\partial_a = L^b_{aj}\dot\partial_b \\ \nabla_{\dot\partial_a}\delta_j = C^i_{ja}\delta_i, \ \ \nabla_{\dot\partial_b}\dot\partial_a = C^d_{ab}\dot\partial_d \end{cases} \tag{2}$$

Definition 3. The h- and v-covariant derivation laws associated to the linear d-connection (2) are defined by

$$\begin{cases} D_i w^{ma}_{nb} = \delta_i w^{ma}_{nb} + L^m_{ki} w^{ka}_{nb} - L^k_{ni} w^{ma}_{kb} + L^a_{di} w^{md}_{nb} - L^d_{bi} w^{ma}_{nd} \\ D_c w^{ma}_{nb} = \dot\partial_c w^{ma}_{nb} + C^m_{kc} w^{ka}_{nb} - C^k_{nc} w^{ma}_{kb} + C^a_{dc} w^{md}_{nb} - C^d_{bc} w^{ma}_{nd}. \end{cases} \tag{3}$$

Proposition 1. *The transformation rules for the coefficients of the linear d-connection are*

$$\begin{aligned} &\bar\partial_m B^k_j - B^k_i \bar L^i_{jm}(\bar x, \bar y) + B^i_j B^n_m L^k_{in}(x,y) = 0 \\ &\bar\partial_m M^a_b - M^a_c \bar L^c_{bm}(\bar x, \bar y) + M^c_b B^n_m L^a_{cn}(x,y) = 0 \\ &B^i_n \bar C^n_{ma}(\bar x, \bar y) = M^c_a B^j_m C^i_{jc}(x,y) \\ &M^a_d \bar C^d_{bc}(\bar x, \bar y) = M^d_b M^f_c C^a_{df}(x,y) \end{aligned} \tag{4}$$

where the coordinate transformations on E have the form

$$\begin{aligned} &x^i = x^i(\bar x), \det(\partial x^i/\partial \bar x^j) \neq 0 \\ &y^a = M^a_b(\bar x)\bar y^b, \det(M^a_b(\bar x)) \neq 0 \end{aligned} \tag{5}$$

and we used the notations

$$B^i_j = \bar\partial_j x^i, \quad \bar\partial_j = \frac{\partial}{\partial \bar x^j}$$

Definition 4. A gauge transformation is a automorphism of E [7,2,3], *locally given by*

$$\begin{aligned} &x^i = X^i(\tilde x), \det(\tilde\partial_j X^i) \neq 0 \\ &y^a = Y^a(\tilde x, \tilde y), \det(Y^a_b) \neq 0, \ \ \dot{\tilde\partial}_c Y^a_b = 0 \end{aligned} \tag{6}$$

where we denoted

$$Y^a_b = \dot{\tilde\partial}_b Y^a, \quad \dot{\tilde\partial}_b = \frac{\partial}{\partial \tilde y^b}, \quad \tilde\partial_j = \frac{\partial}{\partial \tilde x^j}$$

Definition 5. A (generalized [2,3]) gauge tensor field is a field on E, which obeys tensorial rules of transformation relative to (5) and (6); e.g. $\left\{w^{ia}_{jb}\right\}$ obeys

$$\begin{aligned} &B^i_k M^a_c \bar w^{kc}_{jb} = B^\ell_j M^d_b w^{ia}_{\ell d} \\ &X^i_k Y^a_c \tilde w^{kc}_{jb} = X^\ell_j Y^d_b w^{ia}_{\ell d}, \ \ \text{where} \ \ X^i_k = \tilde\partial_k X^i. \end{aligned} \tag{7}$$

Definition 6. A gauge covariant derivation (h-resp. v-) is given by the h- and v-derivation laws in definition 3, which preserves the gauge tensorial character relative to (5), (6).

Proposition 2. *The coefficients of the h- and v-gauge covariant derivations have with respect to* (6) *the transformation laws*

$$
\begin{aligned}
&\tilde{\partial}_m X^k_j - X^k_i \tilde{L}^i_{jm}(\tilde{x},\tilde{y}) + X^i_j X^n_m L^k_{in}(x,y) = 0\\
&\tilde{\partial}_m Y^a_b - Y^a_c \tilde{L}^c_{bm}(\tilde{x},\tilde{y}) + Y^c_b X^n_m L^a_{cn}(x,y) = 0\\
&X^i_n \tilde{C}^n_{ka}(\tilde{x},\tilde{y}) = Y^c_a X^j_k C^i_{jc}(x,y)\\
&Y^a_d \tilde{C}^d_{bc}(\tilde{x},\tilde{y}) = Y^d_b Y^f_c C^a_{df}(x,y)
\end{aligned}
\tag{8}
$$

Remarks.
1. $\{C^i_{ja}\}$ and $\{C^a_{bc}\}$ are gauge tensor fields.

2. The coefficients $\{L^i_{jk}, L^a_{bk}, C^i_{ja}, C^a_{bc}\}$ of the h- and v-gauge covariant derivations (3) are in fact the coefficients of a linear d-connection which satisfies the supplementary rules (8) (gauge linear d-connection).

Proposition 3. *The torsion and the curvature gauge tensor fields of a gauge linear d-connection are given by* [11]

$$
\begin{gathered}
T^i_{jk} = L^i_{[jk]}, \quad R^a_{jk} = -\delta_{[j} N^a_{k]}, \quad P^i_{jc} = C^i_{jc}\\
P^a_{jb} = \dot{\partial}_b N^a_j - L^a_{bj}, \quad S^a_{bc} = C^a_{[bc]}
\end{gathered}
\tag{9}
$$

and respectively

$$
\begin{aligned}
R^i_{jk\ell} &= \delta_{[\ell} L^i_{jk]} + L^h_{j[k} L^i_{h\ell]} + C^i_{ja} R^a_{k\ell},\\
R^a_{bk\ell} &= \delta_{[\ell} L^a_{bk]} + L^c_{b[k} L^a_{c\ell]} + C^a_{bc} R^c_{k\ell},\\
P^i_{jkc} &= \dot{\partial}_c L^i_{jk} - D_k C^i_{jc} + C^i_{jb} P^b_{kc},\\
P^a_{bkc} &= \dot{\partial}_c L^a_{bk} - D_k C^a_{bc} + C^a_{bd} P^d_{kc},\\
S^i_{jbc} &= \dot{\partial}_{[c} C^i_{jb]} + C^h_{j[b} C^i_{hc]}\\
S^a_{bcd} &= \dot{\partial}_{[c} C^a_{bd]} + C^e_{b[c} C^a_{ed]}
\end{aligned}
\tag{9'}
$$

where we used the notation for $[i \ldots j]$*:*

$$
L^h_{k[i} L^s_{hj]} = L^h_{ki} L^s_{hj} - L^h_{kj} L^s_{hi}
$$

Proposition 4. *The following mixed Lagrangian is invariant under* (5) *and* (6) *(i.e. it is a scalar gauge field)*

$$
L = \sum_{i \in I} n_i \cdot L_i, \quad n_i \in \mathbb{R}, \qquad i \in I = \{\overline{1,5}, \overline{11,16}, \overline{21,22}\}
\tag{10}
$$

where

$$
\begin{aligned}
&L_1 = T^i_{jk}T^{jk}_i, \quad L_2 = R^a_{jk}R^{jk}_a, \quad L_3 = P^i_{jc}P^{jc}_i,\ L_4 = P^a_{jb}P^{jb}_a\\
&L_5 = S^a_{bc}S^{bc}_a, \quad L_{21} = R^i_{jkl}R^{jkl}_i,\ L_{22} = S^a_{bcd}S^{bcd}_a,\\
&L_{11} = R^{ij}{}_{ij}, \quad L_{12} = R^a_{bk\ell}R^{bk\ell}_a,\ L_{13} = P^i_{jkc}P^{jkc}_i\\
&L_{14} = P^a_{bkc}P^{bkc}_a,\ L_{15} = S^i_{jbc}S^{jbc}_i,\ L_{16} = S^{ab}{}_{ab}
\end{aligned}
\tag{11}
$$

The proof is computational.

Remark. The Lagrangian L contains, relative to [2], the supplementary terms n_3L_3 and $n_{15}L_{15}$, and the terms $n_{11}L_{11}$ and $n_{13}L_{13}$ are altered (are more general) since the present context doesn't impose the restrictive condition $C^i_{ja} = 0$.

The raising/lowering of the corresponding indices in (11) were performed via the gauge metric tensor fields $\{g_{ij}(x,y)\}$ and $\{h_{ab}(x,y)\}$ [3,2]. Then, introducing the Lagrangian density

$$\mathcal{L} = LG \tag{12}$$

where $G = |\det(g_{ij})|^{1/2} \cdot |\det(h_{ab})|^{1/2}$, we notice that it depends on the gauge fields

$$\phi \in \{N^a_i,\ L^i_{jk},\ L^a_{bk},\ C^i_{ja},\ C^a_{bc},\ g_{ij},\ h_{ab}\} \tag{13}$$

and their derivatives; considering the variational principle [1,3]

$$\delta \int \mathcal{L} dx^n dy^m = 0$$

one can derive the extremum condition of vanishing the Euler–Lagrange derivatives

$$\frac{\delta \mathcal{L}}{\delta \phi} \equiv \frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{L}}{\partial(\partial_j \phi)}\right) + \frac{\partial}{\partial y^a}\left(\frac{\partial \mathcal{L}}{\partial(\dot{\partial}_a \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Theorem 1. *The generalized Einstein–Yang Mills equations associated to the Lagrangian* (10) *for the set of arguments* (13) *are*

$$
\begin{aligned}
\frac{\delta L}{\delta N^a_k} = &-4n_2(D^*_\ell R^{\ell k}_a + P^c_{a\ell}R^{\ell k}_c + \frac{1}{2}T^k_{n\ell}R^{n\ell}_a) -\\
&-2n_4(D^*_b P^{bk}_a + C^c_{ab}P^{bk}_c + \underline{P^k_{nb}P^{nb}_a}) -\\
&-4\underline{n_{21}}[D^*_\ell V^{\ell k}_a + P^c_{a\ell}V^{\ell k}_c + \frac{1}{2}T^k_{n\ell}V^{n\ell}_a - R^{j\ell k}_i(P^i_{j\ell a} + D_\ell P^i_{ja} - P^i_{jb}P^b_{\ell a})] -\\
&-n_{11}[D^*_\ell U^{\ell k}_a + P^c_{a\ell}U^{\ell k}_c + \frac{1}{2}T^k_{n\ell}U^{n\ell}_a - w^{j[lk]}_i(P^i_{j\ell a} + \underline{D_\ell P^i_{ja} - P^i_{jb}P^b_{\ell a}})] +
\end{aligned}
\tag{15.1}
$$

$$+4n_{12}[P^c_{b\ell a}R^{b\ell k}_c + C^c_{ba}\cdot\frac{1}{4}\delta L_{12}/\delta L^c_{bk}]+$$
$$+2\underline{n_{13}}[S^i_{jac}P^{jkc}_i + (D^*_c P^{jkc}_i - P^k_{nc}P^{jnc}_i)P^i_{ja}]+$$
$$+2n_{14}[S^d_{bac}P^{bkc}_d + (D^*_c P^{bkc}_d - P^k_{nc}P^{bnc}_d)C^d_{ba}] = 0,$$

$$\frac{\delta L}{\delta L^i_{jk}} = -4n_1 T^{jk}_i + 4\underline{n_{21}}(D^*_\ell R^{jk\ell}_i - \frac{1}{2}T^k_{n\ell}R^{jn\ell}_i) \tag{15.2}$$
$$+n_{11}(D^*_\ell w^{j[k\ell]}_i - \frac{1}{2}T^k_{n\ell}w^{j[n\ell]}_i) + 2\underline{n_{13}}(D^*_c P^{jkc}_i - P^k_{nc}P^{jnc}_i) = 0,$$

$$\frac{\delta L}{\delta L^a_{bk}} = -2n_4 P^{bk}_a + 4n_{12}(D^*_\ell R^{bk\ell}_a - \frac{1}{2}T^k_{n\ell}R^{bn\ell}_a) \tag{15.3}$$
$$+2n_{14}(D^*_c P^{bkc}_a - \underline{P^k_{nc}P^{bnc}_a}) = 0,$$

$$\frac{\delta L}{\delta C^i_{ja}} = -2\underline{n_3}P^{ja}_i + 2\underline{n_{21}}R^a_{k\ell}R^{jk\ell}_i - n_{11}R^{ja}{}_i -$$
$$-2n_{13}(D^*_\ell P^{j\ell a}_i - P^a_{b\ell}P^{j\ell b}_i)- \tag{15.4}$$
$$-4\underline{n_{15}}(D^*_d S^{jda}_i - \frac{1}{2}S^a_{bd}S^{jdb}_i) = 0,$$

$$\frac{\delta L}{\delta C^a_{bc}} = -4n_5 S^{bc}_a + 2n_{12}R^c_{k\ell}R^{bk\ell}_a + \underline{4n_{22}}(D^*_d S^{bcd}_a - \frac{1}{2}S^c_{ed}S^{bed}_a)+ \tag{15.5}$$
$$+n_{16}(D^*_d w^{b[cd]}_a - \frac{1}{2}S^c_{ed}w^{b[ed]}_a) - 2n_{14}(D^*_\ell P^{b\ell c}_a - P^c_{d\ell}P^{b\ell d}_a) = 0,$$

$$\frac{\delta L}{\delta g_{ij}} = -\frac{\partial L}{\partial g_{ij}} - \frac{1}{2}g^{ij}L = 0, \tag{15.6}$$

$$\frac{\delta L}{\delta h_{ab}} = -\frac{\partial L}{\partial h_{ab}} - \frac{1}{2}h^{ab}L = 0, \tag{15.7}$$

where we denoted

$$\begin{cases} D^*_\ell = D_\ell + V_\ell, \quad V_\ell = \dfrac{D_\ell G}{G} + T^n_{n\ell} + P^d_{d\ell} \\ D^*_c = D_c + V_c, \quad V_c = \dfrac{D_c G}{G} + P^n_{nc} + S^d_{dc} \end{cases}$$

$$V^{k\ell}_a = C^i_{ja}R^{jk\ell}_i,\ U^{k\ell}_a = C^i_{ja}w^{j[k\ell]}_i$$
$$w^{jk\ell}_i = g^{jk}\delta^\ell_i,\ w^{bcd}_a = h^{bc}\delta^d_a; \qquad \frac{\delta L}{\delta\phi} \equiv \frac{1}{G}\frac{\delta\mathcal{L}}{\delta\phi}.$$

Hint. The Euler–Lagrange derivatives for the elementary Lagrangians (11) computed using the relations

$$D_kG = \delta_kG - G(L^n_{nk} + L^a_{ak})$$
$$D_aG = \dot\partial_aG - G(C^n_{na} + C^d_{da}).$$

give by addition the equations above.

Remark. The underscored terms are new with respect to [2], and the equations in [1,2] can be viewed as a particular case of (15.1)–(15.7). The notations of tensor fields and vertical indices are changed from those used by G.S. Asanov to the corresponding ones used in the papers [8,9,11] in the theory of Finsler spaces. Also, the fact that in [1,2] the coefficients of the nonlinear connection considered in (1) are taken with opposite sign, induce related differences of sign in (15.1)–(15.7).

In the following we consider the quasi-metric case, i.e. the situation in which the gauge metric tensor fields obey

$$D_kg_{ij} = 0, \quad D_kh_{ab} = 0, \quad D_ch_{ab} = 0, \tag{16}$$

and impose for the gauge linear d-connection (2) to be h- and v-symmetrical, i.e. $T^i_{jk} = 0$ and $s^a_{bc} = 0$. Then we can state the following

Theorem 2. *The generalized Einstein–Yang Mills equations in the quasi-metric h- and v-symmetrical case, for the generalized gauge Lagrangian* [2]

$$\begin{aligned} L =& n_1R^a_{jk}R^{jk}_a + n_3P^a_{jb}P^{jb}_a + \ell_1R^{mn}{}_{mn} + \ell_2R^{abk\ell}R_{abk\ell} + \\ &+ \ell_{10}S^{ab}{}_{ab} + \Lambda, \qquad \Lambda \in \mathcal{F}(E) \end{aligned} \tag{17}$$

with respect to the set of arguments

$$\phi \in \{N^a_i, A_{abi} = \frac{1}{2}L_{[ab]i}, C^i_{ja}, g_{ij}, h_{ab}\} \tag{18}$$

are the following

$$\begin{aligned} \frac{\delta L}{\delta N^a_i} =& 4n_1(D^*_mR^{im}_a - P^b_{ka}R^{ik}_b) + 2n_3(D^*_bP^{ib}_a - \underline{P^{nb}_aP^i_{nb}}) + \\ &+ \ell_1[\underline{D^*_jU^{ij}_a + U^{ij}_bP^b_{aj}} - w^{nji}_k(-\dot\partial_aL^k_{nj} + V_ng^{km}\dot\partial_ag_{mj})] + \\ &+ 4\ell_2R^{bcki}P_{bcka} + \frac{1}{2}C_{cba}\cdot\frac{\delta L}{\delta A_{cbi}} = 0, \end{aligned} \tag{19.1}$$

$$\begin{aligned} \frac{\delta L}{\delta g_{ij}} =& -\frac{1}{2}g^{ij}L + 2n_1R^{iak}R^j_{ak} + n_3P^{iab}P^j_{ab} + 2\ell_2R_{abk}{}^iR^{abkj} + \\ &+ \ell_1(R^{inj}{}_n + g^{ij}D^*_kV^k - \frac{1}{2}D^{*\{i}V^{j\}} + \frac{1}{2}\underline{P^{\{in}{}_aR^{j\}a}{}_n}) = 0, \end{aligned} \tag{19.2}$$

$$\frac{\delta L}{\delta h_{ab}} = -\frac{1}{2}h^{ab}L - n_1 R^{ka\ell}R^{b}_{k\ell} + \ell_2 D^*_c R^{\{b}_{k\ell}R^{ca\}k\ell} - \frac{1}{2}L^{\{a}_{dj}\frac{\delta L}{\delta A_{db\}j}} +$$
$$+ n_3(P^a_{jc}P^{jcb} - P^a_{jc}P^{jbc} - \frac{1}{2}D^*_j P^{j\{ab\}}) + \tag{19.3}$$
$$+ \ell_{10}(S^{acb}{}_c + h^{ab}D^*_c V^c - \frac{1}{2}D^{*\{a}V^{b\}}) = 0,$$

$$\frac{\delta L}{\delta A_{abi}} = n_3 P^{i[ba]} - 4\ell_2 D^*_n R^{abni} = 0, \tag{19.4}$$

$$\frac{\delta L}{\delta C^i_{ja}} = \ell_1 R_i{}^{aj} = 0, \tag{19.5}$$

where we denoted

$$\begin{cases} D^*_\ell = D_\ell + V_\ell, \quad V_\ell = P^d_{d\ell} \\ D^*_a = D_a + V_a, \quad V_a = \dot\partial_a \ln\sqrt{|\det(g_{ij})|} \end{cases}$$

$$U^{jk}_a = C_{nma}w^{nmjk}, \quad w^{nmjk} = g^{n[j}g^{mk]}$$
$$\delta L/\delta\phi = \frac{1}{G}\frac{\delta\mathcal{L}}{\delta\phi}; \quad \tau_{\{ij\}} = \tau_{ij} + \tau_{ji}; \quad \tau_{[ij]} = \tau_{ij} - \tau_{ji}\,.$$

Hint. The same procedure as in Theorem 1 can be applied, using the relations [2,11]:

$$L_{abi} = h_{bs}L^s_{ai} = A_{abi} + \frac{1}{2}\delta_i h_{ab}$$
$$C^a_{bc} = \frac{1}{2}h^{ad}(\dot\partial_{\{b}h_{dc\}} - \dot\partial_d h_{bc}).$$

Remark. As in theorem 1, the vanishing of the underscored terms in (19.1)–(19.5) give, as a particular case, the corresponding equations in [2].

The attempt of solving the equations in theorem 2 leads to the following results

Theorem 3. *The generalized Einstein–Yang Mills equations* (19.1)–(19.5) *admit the solution*

$$\{N^a_i, g_{ij}, h_{ab}, A_{abk}, C^i_{ja}\} \tag{20}$$

given by

$$
\begin{aligned}
&N_i^a = -\frac{C_i y^a}{2C}, C \in \mathcal{F}(M), \quad D_k^* \left(\frac{C_k}{C}\right) \neq 0 \\
&g_{ij} = e^{\lambda(v)} \cdot \tau_{ij}(x), \quad v \in \mathcal{F}(E) \\
&h_{ab} = \gamma_{ab}(x) + b(x,y) y_a y_b, \quad \textit{with} \;\; y_a \equiv \gamma_{ab} y^b, b(x,y) = \frac{1}{v^2} \\
&A_{abk} = 0 \\
&C_{ja}^i \;\; \textit{satisfying:} \;\; D_j U_a^{ij} = \eta_j U_a^{ij}, \; U_a^{ij} \equiv C_{nma} g^{n[i} g^{mj]}
\end{aligned} \tag{21}
$$

where $\tau_{ij}(x)$ satisfies the Einstein equations of Riemannian type

$$
E_{ij} = m[\tau_{ij}(\alpha p + \nabla_k \alpha^k) - q^{\alpha_i \alpha_j} - \frac{1}{2} \nabla_{\{i} \alpha_{j\}}] \tag{22}
$$

and (21) *are subject to the following conditions*

$$
\delta_k \gamma_{ab} = 0, \quad v = [C(x) - y^2]^{-1/2} \tag{23}
$$

$$
\lambda(x,y) = \frac{2}{n} \ln \left(v \sqrt{\frac{k}{C(x)}} \right), \quad \textit{with} \;\; k \in \mathbb{R}_+^*, \; C(x) > y^2, \; (y^2 \equiv y_a y^a)
$$

$$
\eta \equiv d \left[\ln(|C(x)|^{(1-m)/2}) \right], \; \Lambda = -m(m-1)\ell_{10}/C(x).
$$

In (21)–(23) ϱ_{ij}, ϱ *and* ∇_k *are the Ricci tensor field, the scalar curvature and the covariant derivative associated to* τ_{ij}, *and we used the notations*

$$
\begin{cases}
C_i = \dfrac{\partial C}{\partial x^i}, \quad \alpha_i = -\dfrac{C_i}{2C}, \quad \alpha = \alpha_i \alpha_j \gamma^{ij} \\
E_{ij} = \varrho_{ij} - \dfrac{1}{2} \varrho \tau_{ij} \quad \textit{(the Einstein tensor field)} \\
p = \dfrac{m(3-n)}{2(n-2)}, \quad q = \dfrac{m}{n-2}
\end{cases}
$$

Proof. The equations (19.1)–(19.5) have a general form; it becomes possible to search for solutions of the family (22) of the form (23)–(23') under additional simplifying assumptions, namely

(A1) $\delta_k h_{ab} = 0$

(A2) $b = \dfrac{1}{v^2} = b(x,z)$, with $z = y^2$

(A3) the differential equation of Riccati type obtained from $\dfrac{\delta L}{\delta h_{ab}} = 0$, to become one of Bernoulli type

(A4) $\delta_k \lambda = 0$, $\delta_k \gamma_{ab} = 0$.

The cosmologycal constant Λ is obtained from $\dfrac{\delta L}{\delta g_{ij}} = 0$, equation which provides the classical Einstein equations in (24). The conditions (A1)–(A4) yield to the form of the class of solutions stated in the theorem.

For the case when ξ is the tangent bundle of M, and $L^n = (M, g_{ij}(x,y))$ is a structure of a generalised Lagrange space [11] endowed with the non-linear connection $\{N_i^a(x,y)\}$

$$N_i^a = \left\{ \begin{matrix} a \\ ij \end{matrix} \right\} y^j, \ \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2}\gamma^{is}\left(\frac{\partial \gamma_{sj}}{\partial x^k} + \frac{\partial \gamma_{sk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^s}\right) \tag{24}$$

and the fundamental tensor field

$$g_{ij}(x,y) = \gamma_{ij}(x) + \frac{1}{c^2} y_i y_j; \quad y_i = \gamma_{is} y^s, \quad c > 0, \tag{25}$$

where $\{\gamma_{ij}(x)\}$ is a Riemannian metric on M, we consider the N-lift of g_{ij} to TM ([11])

$$G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ab}(x,y)\delta y^a \otimes \delta y^b. \tag{26}$$

Then, for the case of the canonic metrical h- and v-symmetric linear d-connection, we obtain the coefficients

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} g^{in}(\delta_{\{j} g_{nk\}} - \delta_n g_{jk}) \\ C^a_{bc} &= \frac{1}{2} h^{ad}(\dot\partial_{\{b} h_{dc\}} - \dot\partial_d h_{bc}) \\ \text{with} \ \ h_{ab} &\equiv g_{ab} = g_{ij}\delta^i_a \delta^j_b, \ \ \text{and} \\ L^a_{bk} &= L^i_{jk}\delta^a_i \delta^j_b, \ C^i_{jc} = C^a_{bc}\delta^i_a \delta^b_j \end{aligned} \tag{27}$$

and can formulate the following

Theorem 4. *If L^n is locally Minkowskian, then the gauge fields* (13) *given by* (24), (25), (27) *provide solutions for the generalized Einstein–Yang Mills equations* (19.1)–(19.5) *iff $n = 2$ and*

$$\Lambda = 6\ell_{10}/(1 + 3y^2)$$

where $y^2 = \gamma_{ij}y^i y^j$ and $\left\{ \begin{matrix} a \\ ij \end{matrix} \right\}$ are the Christoffel coefficients for $\gamma_{ij}(x)$, (see (24)*).*

Remark. The vanishing of the cosmological constant Λ would infer that (19.1)–(19.5) have no solution of the given form, unless $\ell_{10} = 0$.

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References

[1] A.K. Aringazin and G.S. Asanov, Problems of Finslerian Theory of Gauge Fields and Gravitation, *Rep. on Math. Phys.* **25** (1988), 35–93.
[2] G.S. Asanov, Fibered Generalization of the Gauge Field Theory, Finslerian and Jet Gauge Fields, *Univ. Moskow*, 1989.
[3] G.S. Asanov and S.F. Ponomarenko, Finsler Bundles over Space-Time, Associated Gauge Fields and Connections, *Nauka Ed.*, *Chishinew*, 1989.
[4] V. Balan, Generalized Einstein – Yang Mills Equations in Vector Bundles, *Bull. Soc. Sci. Mat. Rom.* **36** (1992), 17-22.
[5] V. Balan, Palatini Methods for Vector Bundles, Proc. of the XXII-nd Nat. Conf. of Geometry and Topology, *Bull. of the Polytechnic Inst. of Bucharest* **53** (1991).
[6] H.J. Duff, B.E.W. Nilson and C.N. Pope, Kaluza-Klein Supergravity, *Physical Reports* **130** (1986), 1–130.
[7] K.S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, *Interscience Publishers*, *New York–London*, I, 1963, II, 1969.
[8] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, *Kaisheisha Press*, *Tokyo*, 1986.
[9] R. Miron, Introduction to the Theory of Finsler Spaces, *Proc. of the Nat. Sem. of Finsler Spaces, Brasov feb.* (1980), *Timosoara* (1981), pp. 131–183.
[10] R. Miron, Vector Bundles. Finsler Geometry, *Proc. of the Nat. Sem. of Finsler Spaces, Brasov feb. (1982)*, pp. 147–188.
[11] R. Miron and M. Anastasiei, Vector Bundles. Lagrange Spaces, Applications to the Theory of Relativity, *Academy Ed.*, *Bucharest*, 1987.
[12] A. Morimoto, Prolongation of Connections to Bundles of Infinitely Near Points, *Jour. of Diff. Geom.* **11** (1976), 479–498.
[13] H. Rund, The Differential Geometry of Finsler Spaces, *Nauka Ed.*, *Moskow*, 1981.

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