

A new proof on Lévy's random domain

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Dedicated to Professor Lajos Tamásy on his 70th birthday

Abstract. In this note I give a new analytical proof of LÉVY's result [3], [6] on the distribution of the directed random domain $\int[\xi_1(s)d\xi_2 - \xi_2(s)d\xi_1]$ for complex AR (autoregression) processes. In LÉVY's book [6] this statement is given for Brownian motion processes. In statistical literature this result is known, but the proof of it is very sophisticated (see [2], [3], [5], [7], [9]).

1. In the statistical inference of stochastic processes with continuous time we have only a few examples for exact distributions of statistics. Even these results are connected with the Brownian motion (see, e.g., [6], [7]). Using NOVIKOV's method ([9], [8]) I gave a new proof for the estimator of the periodical component of a complex valued AR process [3]. This theorem was stated by KOLMOGOROV in the early sixties [5]. The so called damping parameter was discussed in [1]. The real valued AR process was investigated later [4], [2].

Let the complex valued process $\xi(t) = \xi_1(t) + i\xi_2(t)$ be the solution of the following differential (stochastic) equation

$$(1) \quad d\xi(t) = -\gamma\xi(t)dt + dw(t),$$

where

$$\gamma = \lambda - i\omega, \quad \lambda, \omega > 0, \quad w(t) = w_1(t) + iw_2(t),$$

where $w_1(t)$, $w_2(t)$ are independent Brownian motion processes (standard) $\mathbb{E} w_j(t) = 0$, $\mathbb{E}(w_j(t))^2 = t$ ($j = 1, 2$). If $\xi(t)$ is the stationary solution of

(1) then

$$\mathbb{E}\xi(t)\overline{\xi(t+\tau)} = \frac{1}{\lambda}e^{-\lambda\tau-i\omega\tau}, \quad \tau > 0.$$

Let the process $\theta(t)$ be defined by

$$\xi(t) = |\xi(t)|e^{i\theta(t)}.$$

We assume that $\eta(t)$ is another complex valued process with different (λ_2, ω_2) parameters, and with the same Brownian motion $w(t)$

$$(2) \quad d\eta(t) = -\gamma_2\eta(t)dt + dw(t), \quad \gamma_2 = \lambda_2 - i\omega_2, \quad \lambda_2, \omega_2 > 0.$$

Let $\mathbb{P}_{\lambda, \omega}$, $\mathbb{P}_{\lambda_2, \omega_2}$ be the measures generated by the processes $\xi(t)$ and $\eta(t)$, respectively. Then the Radon–Nikodym derivative on the interval $[0, t]$ has the following form

$$(3) \quad \frac{d\mathbb{P}_{\lambda, \omega}}{d\mathbb{P}_{\lambda_2, \omega_2}}(\xi) = \frac{\lambda}{\lambda_2} \exp \left\{ -\frac{\lambda^2 + \omega^2 - (\lambda_2^2 + \omega_2^2)}{2} s_\xi^2(t) - \right. \\ \left. - \frac{\lambda - \lambda_2}{2} [|\xi(0)|^2 + |\xi(t)|^2] + (\lambda - \lambda_2)t + (\omega - \omega_2)r_\xi(t) \right\},$$

where we use the following functionals of the process $\xi(t)$

$$(4) \quad s_\xi^2(t) = \int_0^t [\xi_1^2(s) + \xi_2^2(s)] ds = \int_0^t |\xi(s)|^2 ds, \\ r_\xi(t) = \int_0^t [\xi_1 d\xi_2 - \xi_2 d\xi_1] = \int_0^t |\xi(s)|^2 d\theta(s).$$

Statement. *If $\xi(t)$ is the solution of (1) and it is stationary, $\mathbb{E}\xi(t) = 0$, $\mathbb{D}^2\xi_i(t) = \frac{1}{2\lambda}$ ($i = 1, 2$), then, $(\Lambda = \sqrt{\lambda^2 + 2a})$,*

$$(5) \quad \mathbb{E} \exp\{-as_\xi^2(t)\} = \\ = 4\lambda\Lambda e^{\lambda t} [(\lambda + \Lambda)^2 \exp(\Lambda t) - (\lambda - \Lambda)^2 \exp(-\Lambda t)]^{-1}.$$

If $\xi(0) = x + iy$ and $\xi(t)$ is the solution of (1) then we have

$$(6) \quad \mathbb{E} \exp\{-as_\xi^2(t)\} = \\ = \exp \left\{ \lambda t - (x^2 + y^2)a[\Lambda \coth \Lambda t + \lambda]^{-1} \right\} \left[\cosh \Lambda t + \frac{\lambda}{\Lambda} \sinh \Lambda t \right]^{-1}.$$

The proof can be seen in [2], [7], [8], [9] (formulae (3.3.5), (3.6.5), (4.2.15), (4.3.9) in [2]). Our main result is the following

Theorem. *If $\xi(t)$ is the stationary solution of (1) then the statistic*

$$(r_\xi(t) - \omega \cdot s_\xi^2(t)) (s_\xi^2(t))^{-1/2} = \zeta(t)$$

has a standard normal $(0,1)$ distribution.

PROOF. Let us consider the expectation

$$\begin{aligned} v(c) &= \mathbb{E}_{\lambda, \omega} \exp \{ -(cr_\xi(t) - c\omega s_\xi^2(t)) \} = \\ &= \mathbb{E} \exp \left\{ -c \int_0^t |\xi(s)|^2 d\theta(s) + c\omega \int_0^t |\xi(s)|^2 ds \right\}, \end{aligned}$$

which can be rewritten by the help of the Radon–Nikodym transformation, assuming $\lambda_2 = \lambda$, $\omega_2 = \omega - c$, in the form

$$\begin{aligned} v(c) &= \mathbb{E} \exp \left\{ -\frac{\omega^2 - (\omega - c)^2 - 2c\omega}{2} \int_0^t |\eta(s)|^2 ds \right\} = \\ &= \mathbb{E} \exp \left\{ \frac{c^2}{2} \int_0^t |\eta(s)|^2 ds \right\}. \end{aligned}$$

Under the condition $s_\eta^2 = \text{const} = \sigma^2$ we get

$$v(c) = \exp \left\{ \frac{c^2 \sigma^2}{2} \right\},$$

and this is the moment generating function of a normally distributed random variable and gives that ζ has a normal distribution with $(0,1)$ parameters, independently of $s_\eta^2(t)$. Then the unconditional distribution of the random variable ζ is also normal. This proves the theorem.

2. Now we consider the second order AR process with complex roots. We assume that $\xi(t)$ is the solution of the following stochastic differential equation, where $w(t)$ is a standard Brownian motion

$$(7) \quad d\xi'(t) + [A_1 \xi'(t) + A_2 \xi(t)]dt = dw(t).$$

We take the process $\eta(t)$ with the same brownian motion and different parameters, which will be defined later

$$(8) \quad d\eta'(t) + [a_1 \xi'(t) + a_2 \xi(t)]dt = dw(t).$$

The Radon–Nikodym derivative has the following form, in the stationary case, (see (4.5.6) or (4.7.45a) in [2])

$$(9) \quad \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(x(t)) = \frac{A_1\sqrt{A_2}}{a_1\sqrt{a_2}} \exp \left\{ -\frac{A_2^2 - a_2^2}{2} \int_0^t x^2(s)ds + \right. \\ \left. + (A_2 - a_2) \int_0^t (x'(s))^2 ds - \frac{A_1^2 - a_1^2}{2} \int_0^t (x'(s))^2 ds + \right. \\ \left. + \frac{A_1 - a_1}{2} t - \frac{A_1 - a_1}{2} [(x'(t))^2 + (x'(0))^2] - \right. \\ \left. - \frac{A_2 - a_2}{2} [x(t)x'(t) - x(0)x'(0)] + \right. \\ \left. + \frac{A_1 A_2 - a_1 a_2}{2} [(x(t))^2 + (x(0))^2] \right\}$$

If¹ A_1 is the only unknown parameter, then the sufficient statistic is

$$\int_0^t (x'(s))^2 ds, \quad (x'(t))^2 + (x'(0))^2, \quad (x(t))^2 + (x(0))^2.$$

For A_2 the sufficient statistic is

$$\int_0^t (x(s))^2 ds, \quad \int_0^t (x'(s))^2 ds, \quad x(0)x'(0) - x(t)x'(t), \quad (x(t))^2 + (x(0))^2.$$

a) The moment generating function of the functional $\int_0^t (x'(s))^2 ds$ can be calculated in the following way. Let be $a_2 = A_2$ and $-2c = A_1^2 - a_1^2$, then in the usual way we get (using (7)–(9))

$$(10) \quad \mathbb{E}_{A_1, A_2} \exp \left\{ -c \int_0^t (x'(s))^2 ds \right\} = \mathbb{E} \exp \left\{ -c \int_0^t (\xi'(s))^2 ds \right\} = \\ = \frac{A_1}{a_1} \mathbb{E} \exp \left\{ -c \int_0^t (\eta'(s))^2 ds - \frac{A_1^2 - a_1^2}{2} \int_0^t (\eta'(s))^2 ds + \frac{A_1 - a_1}{2} t - \right. \\ \left. - A_2 \frac{A_1 - a_1}{2} [\eta^2(t) + \eta^2(0)] - \frac{A_1 - a_1}{2} [(\eta'(t))^2 + (\eta'(0))^2] \right\}.$$

As the integral functionals disappear in the exponent we get

$$\mathbb{E} \exp \left\{ -c \int_0^t (\xi'(s))^2 ds \right\} = \frac{A_1}{a_1} \exp \left\{ \frac{A_1 - a_1}{2} t \right\} \cdot \\ \cdot \mathbb{E} \exp \left\{ -\frac{A_1 - a_1}{2} [\eta'(0)^2 + A_2 \eta(0)^2 + \eta'(t)^2 + A_2 \eta(t)^2] \right\}.$$

¹We assume that the roots of the characteristic equation $x^2 + A_1 x + A_2 = 0$ are complex.

The “natural parameters” are: $\lambda = \frac{A_1}{2}$, $\omega = \sqrt{A_2 - \frac{A_1^2}{4}}$, $(x_{1,2} = -\lambda \pm i\omega)$.

The expectation can be calculated as the random variables $\eta'(0)^2 + A_2\eta(0)^2$ and $\eta'(t)^2 + A_2\eta(t)^2$ are Chi-square distributed (but correlated).

b) The moment generating function of the functional $\int_0^t \xi(s)^2 ds$ depends on the functional $\int_0^t (\xi(s)')^2 ds$, too. Let us put $a_1 = A_1$ and $-c = \frac{1}{2}(A_2 + a_2)(A_2 - a_2)$. Then we have

$$(11) \quad \mathbb{E}_{A_1, A_2} \exp \left\{ -c \int_0^t x^2(s) ds - \frac{A_2 - a_2}{2} \int_0^t (x'(s))^2 ds \right\} \\ = \mathbb{E} \exp \left\{ -c \int_0^t (\xi(s))^2 ds - \frac{A_2 - a_2}{2} \int_0^t (\xi'(s))^2 ds \right\}$$

and using (7)–(9)

$$(12) \quad \mathbb{E} \exp \left\{ -c \int_0^t (\xi(s))^2 ds - \frac{A_2 - a_2}{2} \int_0^t (\xi'(s))^2 ds \right\} = \\ = \sqrt{\frac{A_2}{a_2}} \mathbb{E} \exp \left\{ -c \int_0^t (\eta(s))^2 ds - \frac{A_2 - a_2}{2} \int_0^t (\eta'(s))^2 ds - \right. \\ \left. - \frac{A_2 - a_2}{2} \left[(A_2 + a_2) \int_0^t (\eta(s))^2 ds - \int_0^t (\eta'(s))^2 ds + \eta(t)\eta'(t) - \right. \right. \\ \left. \left. - \eta(0)\eta'(0) + A_1\eta^2(t) + A_1\eta^2(0) \right] \right\} = \\ = \sqrt{\frac{A_2}{a_2}} \mathbb{E} \exp \left\{ -\frac{A_2 - a_2}{2} [\eta(t)\eta'(t) - \eta(0)\eta'(0) + A_1(\eta^2(t) + \eta^2(0))] \right\}.$$

In this expectation we have again a quadratic form of normally distributed random variables, which can be calculated. This form does not give that the estimator of ω is normal.

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