

## On the exponent of the group of normalized units of a modular group algebras

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*Dedicated to Professor Lajos Tamássy on his 70th birthday*

Let  $G$  be a finite  $p$ -group and  $K$  a finite field of characteristic  $p$ . The group of units of  $KG$  is denoted by  $U(KG)$ . It is easy to show that  $U(KG) = U(K) \times V(KG)$  where

$$V(KG) = \left\{ \sum_{g \in G} \alpha_g g \in KG \mid \sum_{g \in G} \alpha_g = 1, \alpha_g \in K \right\}$$

is the group of normalized units.  $V(KG)$  is a  $p$ -group and  $|V(KG)| = p^{r(|G|-1)}$  where  $|K| = p^r$ . Clearly  $V(KG)$  is a normal Sylow  $p$ -subgroup in  $U(KG)$ .

In general the problem of determining the exponent of  $V(KG)$  is open, the first partial result was obtained by Z. PATAY [4] and A. SHALEV [5]. It is an interesting and important problem.

Since  $G$  is embedded in  $V(KG)$ , we obviously have

$$\exp(V(KG)) \geq \exp(G),$$

but usually the exponent of  $V(KG)$  is much larger. Indeed, by the result of COLEMAN and PASSMAN [4] if  $G$  is non-abelian and  $p \neq 2$  then the wreath product  $C_p wr C_p$  is involved in  $V(KG)$ , and we get

$$\exp(V(KG)) \geq p^2.$$

Moreover, it turns out that for every  $p \neq 2$  there exists a sequence  $\{G_m\}_{m \geq 1}$  of finite groups of exponent  $p$ , such that  $\exp(V(KG_m)) \rightarrow \infty$

[5]. For example, we may choose  $G_m$  as the free nilpotent group of class 2 and exponent  $p$  on  $m$  generators. This is a consequence of the theorem of PASSMAN [2] on polynomial identities in group rings of characteristic  $p$ . Therefore one cannot expect general inequalities of the form  $\exp(V(KG)) \leq f(\exp(G))$  for a fixed function  $f : N \rightarrow N$ . On the other hand,  $\exp(V(KG))$  can be small for arbitrary large  $G$ . Thus, for example, if  $G$  is abelian, then  $\exp(V(KG)) = \exp(G)$ . ANER SHALEV [5] proved that if  $p \geq 7$  and  $\exp(G)^3 > |G|$ , then  $G$  and  $V(KG)$  have the same exponent. Our main problem is identify situations when  $\exp(V(KG))$  is finite and close to  $\exp(G)$ .

If  $1 + A(KG) = V(KG)$  and  $V(KG)$  has a finite exponent then by Zelmanov's theorem and by theorem 2.12 [1]  $G$  is a locally finite  $p$ -group and  $K$  is a field of characteristic  $p$ .

We proved the following results:

**Theorem 1.** *Let  $G$  be a locally finite  $p$ -group. Then  $V(KG)$  has finite exponent if and only if  $G$  has finite exponent and there exists a normal subgroup  $L$  of finite index in  $G$  such that the commutator subgroup of  $L$  is finite.*

PROOF. Let  $p^n$  be the exponent of  $V(KG)$ . Clearly  $V(KG) = 1 + A(KG)$  and for every  $x \in KG$  there exists such  $\alpha \in K$  that  $\alpha + x \in V(KG)$ . Then the Lie product  $((\alpha + x)^{p^n}, y)$  coincides with  $(x^{p^n}, y)$  and  $KG$  satisfies the polynomial identity  $(x^{p^n}, y) = 0$ , where  $y \in KG$ . By Passman's theorem [2] we know the structure of the group  $G$ .

Now suppose that  $G$  has a normal subgroup  $L$  such that  $|G/L| = p^m$  and the commutator subgroup  $L'$  has order  $p^t$ . Let  $I(L')$  be an ideal of  $KG$  generated by the elements  $g - 1$  ( $g \in L'$ ). It is well-known that  $I(L')$  nilpotent and

$$V(KG)/(I(L') + 1) \cong V(K(G/L')).$$

Since  $1 + I(L')$  is a subgroup of finite exponent without loss of generality we can assume that  $L$  is abelian. Let  $g_1, \dots, g_{p^m}$  be representatives of the distinct cosets of  $G$  modules  $L$ . If  $x \in V(KG)$ , then there exist elements  $x_i$  in  $KL$  such that

$$x = x_1 g_1 + x_2 g_2 + \dots + x_{p^m} g_{p^m}.$$

Every element  $x_i$  has finite  $G$ -orbit and the order of each orbit is less than  $|G : L|$ . If  $x_i = \sum_{g \in L} \alpha_g^i g$  and  $\chi(x_i) = \sum_{g \in L} \alpha_g^i$ , then  $(x_i - \chi(x_i))^{p^k} = 0$  where  $p^k$  is the exponent of  $L$ . Since

$$x = 1 + (x_1 - \chi(x_1))g_1 + \dots + (x_{p^m} - \chi(x_{p^m}))g_{p^m},$$

we have that  $x^{p^{k+m}} = 1$  which proves the theorem.

**Lemma 1.** *Let  $G$  be such group that*

$$(1) \quad [a^{p^n}, b] = [a, b]^{p^n} c^{p^n}$$

*for all  $a, b \in G$ , where  $c$  is in the commutator group of the group generated by  $a$  and  $b$ . If  $w \in I(G')^{p^{n-1}}$  and  $x_i \in \{w, g^{p^n} \mid g \in G\}$  then the Lie product*

$$(2) \quad (x_1 x_2 \cdots x_k, x_{k+1} \cdots x_p) \in I(G')^{p^n}.$$

PROOF. The following identities hold in  $KG$

$$(3) \quad (uv, w) = (u, v)v + u(v, w),$$

where  $u, v, w \in KG$ . If  $x_i = w$  for all  $i$  than the statment of lemma is trivial.

Suppose that the first  $x_j = g^{p^n}$  and  $x_{j-1} = \cdots = x_1 = w$ . Applying (3), we conclude that

$$\begin{aligned} (x_1 \cdots x_{j-1} g^{p^n} \cdots x_k, x_{k+1} \cdots x_{p^n}) &= (w^{j-1}, x_{k+1} \cdots x_{p^n}) g^{p^n} x_{j+1} \cdots x_k + \\ &\quad + w^{j-1} (g^{p^n} x_{j+1} \cdots x_k, x_{k+1} \cdots x_{p^n}) = \\ &= (w^{j-1}, x_{k+1} \cdots x_p) g^{p^n} x_{j+1} \cdots x_k + w^{j-1} (g^{p^n}, x_{k+1} \cdots x_{p^n}) x_{j+1} \cdots x_k + \\ &\quad + w^{j-1} g^{p^n} (x_{j+1} \cdots x_k, x_{k+1} \cdots x_{p^n}). \end{aligned}$$

Let

$$x_{j+1} \cdots x_{p^n} = \sum_{h \in G} \alpha_h h.$$

Then

$$\begin{aligned} (g^{p^n}, x_{k+1} \cdots x_p) &= \sum_{h \in G} \alpha_h (g^{p^n} h - h g^{p^n}) = \\ &= \sum_{h \in G} \alpha_h g^{p^n} h (1 - [g^{p^n}, h]) = \sum_{h \in G} \alpha_h g^{p^n} h (1 - [g, h]^{p^n} c^{p^n}) \in I(G')^{p^n}. \end{aligned}$$

If  $x_k = w$  ( $k = j+1, \dots, p^n$ ), then the proof is complete; otherwise the argument may be repeated until we see that all relevant commutator are in  $I(G')^{p^n}$ .

**Theorem 2.** *Let  $G$  be a finite  $p$ -group with cyclic commutator subgroup  $G'$ . If  $\exp(G) = \exp(G')$  then  $\exp(V(KG)) = p \cdot \exp(G)$ , and if  $\exp(G) \neq \exp(G')$  then  $\exp(V(KG)) = \exp(G)$ .*

PROOF. Let  $G' = \langle c \rangle$  be the cyclic commutator subgroup of order  $p^n$ . The lower central series of  $G$  will be denoted by

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_s(G).$$

It is well-known the following Hall's collection formula [6]

$$(4) \quad [a^{p^{n-1}}, b] \equiv [a, b]^{p^{n-1}} \bmod (\gamma_2(G))^{p^{n-1}} \prod_{1 \leq r < n} (\gamma_{p^r}(H))^{p^{n-1-r}},$$

where  $H$  is the subgroup generated by elements  $a, [a, b]$ . Because  $\gamma_2(G) = \langle c \rangle$  and  $G$  is nilpotent this formula implies that

$$(5) \quad [a^{p^{n-1}}, b] = [a, b]^{p^{n-1}} c^{kp^{n-1}}$$

for all  $a, b \in G$  and  $k$  depends on  $a, b$ . Let

$$x = \sum_{g \in G} \alpha_g g \in V(KG)$$

and  $L(KG) = [KG, KG]$  is the  $K$ -modul generated by Lie product  $(u, v)$  for any  $u, v \in KG$ . Then by Proposition 3.1 [1]

$$x^p = \sum_{g \in G} \alpha_g^p g^p + w \quad (w \in L(KG)).$$

It is clear that  $L(KG)$  is contained in ideal  $I(G')$  of  $KG$  generated by all  $g - 1$  ( $g \in G'$ ).

Let us prove by induction of on the order  $p^n$  of commutator subgroup of  $G$  that

$$(6) \quad x^{p^n} \equiv \sum_{g \in G} \alpha_g^{p^n} g^{p^n} \bmod I(G')^{p^{n-1}}$$

In the case  $n = 1$  it is true by (4). Let  $H$  be a subgroup of order  $p$  of  $G'$  and  $I(H)$  is an ideal of  $KG$  generated by  $h - 1$  ( $h \in H$ ). Then  $KG/I(H) \cong KG/H$ , the commutator subgroup of  $G/H$  has order  $p^{n-1}$  and  $I(H) \subseteq I(G')^{p^{n-1}}$ . Applying the induction hypothesis, we deduce

$$x^{p^{n-1}} + I(H) \equiv \sum_{g \in G} \alpha_g^{p^{n-1}} g^{p^{n-1}} + I(H) \bmod I(G')^{p^{n-2}}/I(H).$$

Because there exists element  $y \in I(G')^{p^{n-2}}$  such that

$$x^{p^{n-1}} = \sum_{g \in G} \alpha_g^{p^{n-1}} g^{p^{n-1}} + y,$$

then

$$x^{p^n} = \sum_{g \in G} \alpha_g^{p^n} g^p + y^p + \sum u_1 u_2 \cdots u_p, \quad u_i \in \{g^{p^{n-1}}, y\},$$

where the sum is taken over all such products  $u_1 u_2 \cdots u_p$ , that not all  $u_i$  equal to  $g^{p^n}$  or  $y$ , and the sum contains the cyclic permutation of each word  $u_1 u_2 \cdots u_p$ . Because  $KG$  has characteristic  $p$  and

$$u_i u_{i+1} \cdots u_p u_1 u_2 \cdots u_{i-1} - u_1 u_2 \cdots u_p = (u_i u_{i+1} \cdots u_p, u_1 u_2 \cdots u_{i-1}),$$

then  $\sum u_1 u_2 \cdots u_p$  may be represent as Lie product

$$v = (u_{i_1} u_{i_2} \cdots u_{i_k}, u_{i_{k+1}} \cdots u_{i_p}).$$

Let for some Lemma 1 we conclude that  $v \in I(G')^{p^n}$ . Therefore (6) is true and this we can use in determination of the exponent of  $V(KG)$ .

Obviously, the element  $g^{p^n}$  is in the centre for all  $g \in G$  and  $I(G')^{p^n} = 0$ . If  $\exp(G) > p^n$  then it follows from (3) that  $\exp(G) = \exp(V(KG))$ .

Assume that  $\exp(G) = \exp(G') = p^n$ . By virtue of (6)  $\exp(V(KG)) \leq p^{n+1}$ . There exist such  $a, b \in G$  that  $c = [a, b]$  and  $c$  has order  $p^n$ .

We now claim the element  $x = 1 + b^{-1}(a - 1)$  has order  $p^{n+1}$ .

Suppose that  $x^{p^n} = 1$ . Then  $b^{-i} a b^i = a c^{k_i}$  and  $[b^{-1}(a - 1)]^{p^n} = 0$  from which it follows that

$$(ac - 1)(ac^{k_2} - 1) \cdots (ac^{k_{p^n-1}} - 1)(a - 1) = 0.$$

By proposition 2.7 [1]

$$(7) \quad (ac - 1)(ac^{k_2} - 1) \cdots (ac^{k_{p^n-1}} - 1) = (1 + a + \cdots + a^{p^n-1})z \quad (z \in K\langle c \rangle).$$

It easily verified from (5) that  $\langle c \rangle \cap \langle a \rangle = 1$  since  $\langle c \rangle$  is a normal subgroup of  $\langle c, a \rangle$  and comparing coefficients of  $a$  and 1 in (7) we get

$$c = z, \quad -1 = z,$$

which is impossible. Therefore we conclude  $\exp(V(KG)) = p^{n+1}$ .

**Theorem 3.** *Let  $G$  be a finite  $p$ -group of nilpotency class two or  $G$  is a finite  $p$ -regular group. Let  $t(G')$  denotes the nilpotency class of the augmentation ideal  $A(KG')$  and  $k$  is the least integer such that  $t(G') \leq p^k$ . Then*

1. if  $p^k < \exp(G)$ , then  $\exp(V(KG)) = \exp(G)$ ;
2. if  $p^k \geq \exp(G)$ , then  $\exp(V(KG)) \leq p^{k+1}$ .

PROOF. If  $G$  has the nilpotency class two then (1) is valid and the conditions the Lemma 1 is satisfied.

Immediate consequence of the definition of  $p$ -regular group we become

$$[x^{p^{n-1}}, y] = [x, y]^{p^{n-1}} c^{p^{n-1}},$$

where  $c$  is in the commutator subgroup. Then the argument of proof of the theorem 2 may be repeated and we get the statement of this theorem.

**Theorem 4.** *Let  $C$  be the center of a 2-group  $G$  which contains an abelian subgroup  $A$  of index two and  $K$  be a field of two elements. Then*

1. *if  $\exp(A/C) < \exp(A)$  then  $\exp(V(KG)) = \exp(G)$ ,*
2. *if  $\exp(A/C) = \exp(A) = \exp(G)$  then  $\exp(V(KG)) = 2 \cdot \exp(G)$ ,*
3. *if  $\exp(A/C) = \exp(A) < \exp(G)$  then  $\exp(V(KG)) = \exp(G)$ .*

PROOF. Clearly there exists such  $b \in G$  that  $G = \langle A, b \rangle$  and  $b^2 \in A$ . Then every element  $x$  of  $V(KG)$  has a unique representation in the form  $x = x_1 + x_2b$ , where  $x_1, x_2 \in KA$ . It is easy to see that the map defined by  $u \rightarrow \bar{u} = b^{-1}ub$  ( $u \in KA$ ) is an involution on  $KG$  and

$$x^2 = x_1^2 + x_2\bar{x}_2b^2 + x_2(x_1 + \bar{x}_1)b$$

The reader can readily verify by induction that

$$(8) \quad x^{2^n} = x_1^{2^n} + (x_2\bar{x}_2)^{2^{n-1}}b^{2^n} + \sum_{i=1}^{n-1} (x_2\bar{x}_2)^{2^{i-1}}(x_1 + \bar{x}_1)^{2^n-2^i}b^{2^i} + x_2(x_1 + \bar{x}_1)^{2^n-1}b.$$

Let  $\exp(A) = 2^m$ ,  $\exp(A/C) = 2^t$  and  $x_1 = \sum_{a \in A} \alpha_a a$ . Then  $x_1 + \bar{x}_1 = \sum_{a \in A} \alpha_a(a + \bar{a})$  and  $(a + \bar{a})^{2^t} = 0$ , because  $(x_1 + \bar{x}_1)^{2^t} = 0$  and

$$(x_2\bar{x}_2)^{2^t} = x_2^{2^{t+1}}.$$

Suppose that  $t < m$ . It is clear that  $2^m - 2^i \geq 2^{m-1}$  for every  $i \leq m-1$ . By (8) we have

$$x^{2^m} = \chi(x_1)^{2^m} + \chi(x_2)^{2^m}b^{2^m},$$

where  $\chi(x_1) = \sum_{a \in A} \alpha_a$ . Suppose that  $t = m$ . Then

$$x^{2^m} = \chi(x_1)^{2^m} + (x_2\bar{x}_2)^{2^{m-1}}b^{2^m}$$

and we conclude

$$(9) \quad x^{2^{m+1}} = \chi(x_1)^{2^{m+1}} + \chi(x_2)^{2^{m+1}}b^{2^{m+1}}.$$

Because  $x \in V(KG)$  we have  $\chi(x_1) + \chi(x_2) = 1$ . If  $\exp(A) < \exp(V)$ , then (9) implies that  $\exp(V(KG)) = \exp(G)$ .

Assume that  $\exp(A/C) = \exp(A) = \exp(G)$ . Clearly there exist cyclic subgroups  $\langle a_1 \rangle, \langle a_2 \rangle$  of order  $2^m$  in  $A$  such that  $\langle ba_1b^{-1} \rangle \cap \langle a_2 \rangle = 1$ . Then one immediately verifies that  $x = 1 + (a_1 + a_2)b$  has order  $2^{m+1}$ . Thus  $\exp(V(KG)) = 2\exp(G)$ , which proves the theorem.

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*(Received February 12, 1993)*