

Lorentzian Para-contact submanifolds

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MATSUMOTO and MIHAI [1] introduced the idea of Lorentzian Para-contact structure and studied its several properties. The purpose of the present paper is to initiate the study of Lorentzian Para-contact submanifolds.

1. Introduction

Let us consider an n -dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞ vector valued linear function φ , a C^∞ vector field ξ and a C^∞ 1-form η and a Lorentzian metric g satisfying

$$(1.1) \quad \varphi^2(V) = V + \eta(V)\xi$$

$$(1.2) \quad \eta(\xi) = -1$$

$$(1.3) \quad g(\varphi U, \varphi V) = g(U, V) + \eta(U)\eta(V)$$

$$(1.4) \quad g(V, \xi) = \eta(V)$$

for arbitrary vector fields U and V , then V_n is called a Lorentzian Para-contact manifold and the structure (φ, ξ, η, g) is called a Lorentzian Para-contact structure.

In a Lorentzian para-contact structure the following hold:

$$(1.5) \quad \varphi\xi = 0, \quad \eta(\varphi V) = 0$$

$$(1.6) \quad \text{rank}(\varphi) = n - 1.$$

A Lorentzian para-contact manifold is called Lorentzian Para-Sasakian manifold if

$$(1.7) \quad (\nabla_U \varphi)(V) = g(U, V)\xi + U\eta(V) + 2\eta(U)\eta(V)$$

and

$$(1.8) \quad \nabla_U \xi = \varphi U$$

where ∇_U denotes the covariant differentiation with respect to g .
Let us put

$$(1.9) \quad \Phi(U, V) = g(\varphi U, V)$$

Then the tensor field Φ is symmetric:

$$(1.10) \quad \Phi(U, V) = \Phi(V, U)$$

and

$$(1.11) \quad \Phi(U, V) = (\nabla_U \eta)(V).$$

Definition 1.1. An Lorentzian para-contact manifold will be called an LP-cosymplectic manifold if

$$(1.12) \quad \nabla_U \varphi = 0$$

Definition 1.2. An Lorentzian Para (LP)-contact manifold will be called an LP-nearly cosymplectic manifold if

$$(1.13) \quad (a) \quad (\nabla_U \varphi)(U) = 0 \iff (\nabla_U \varphi)(V) + (\nabla_V \varphi)(U) = 0.$$

It can be easily seen that on an LP-Cosymplectic manifold $\nabla_U \xi = 0$.

Theorem 1.1. *The Lorentzian para-contact structure on V_n is not unique.*

PROOF. Let (φ, ξ, η, g) be a Lorentzian Para-contact structure on V_n . Let ξ' be a nonzero vector field nowhere in the ξ -direction, then we have a non-singular tensor field μ of type $(1, 1)$ such that $\mu\xi' = \xi$. If we define a tensor field φ' and a 1-form η' by $\mu\varphi'U = \varphi\mu U$, $\eta'(U) = \eta(\mu U)$, then we have

$$\mu\varphi'^2U = \varphi\mu\varphi'U = \varphi^2\mu U = \mu U + \eta(\mu U) = \mu(U + \eta'(U)\xi)$$

yielding

$$\varphi'^2U = U + \eta'(U)\xi'.$$

Let us define a metric tensor g' by $g'(U, V) = g(\mu U, \mu V)$.

Then $g'(\varphi'U, \varphi'V) = g(\mu\varphi'U, \mu\varphi'V) = g'(U, V) + \eta'(U)\eta'(V)$.

Also $g'(\xi', U) = \eta'(U)$.

Thus $(\varphi', \xi', \eta', g')$ is another Lorentzian Para-contact structure.

Theorem 1.2. *On a nearly LP-Cosymplectic manifold $\nabla_U \xi = 0$.*

PROOF. Equation (1.13) (b) is equivalent to

$$(1.14) \quad (\nabla_U \Phi)(V, W) = (\nabla_V \Phi)(U, W) = 0.$$

Equations (1.9) and (1.14) give

$$(1.15) \quad (\nabla_U \eta)(\varphi V) + (\nabla_V \eta)(\varphi U) = 0.$$

Putting ξ for U in (1.14) and using in (1.15) we get $\nabla_U \xi = 0$.

2. Lorentzian Para-contact submanifold

Let V_{2m-1} be a submanifold of V_{2m+1} with the inclusion map $b : V_{2m-1} \rightarrow V_{2m+1}$ such that $p \in V_{2m-1}$ goes to $bp \in V_{2m+1}$. The map b induces a linear transformation (Jacobian map) $b_* : T_{(2m-1)} \rightarrow T_{(2m+1)}$ where T_{2m-1} is the tangent space to V_{2m-1} at a point p and T_{2m+1} is the tangent space to V_{2m+1} at a point bp such that

$$(X \text{ in } V_{2m-1} \text{ at } p) \rightarrow (b_*X \text{ in } V_{2m+1} \text{ at } bp)$$

Agreement 2.1. In what follows the equations containing X, Y, Z hold for arbitrary vector fields X, Y, Z in V_{2m-1} .

Let M, N be mutually orthogonal unit vectors normal to V_{2m-1} . If $\tilde{g}$ is an induced metric tensor in V_{2m-1} , we have

$$(2.1) \quad \begin{aligned} (a) \quad & g(b_*X, b_*Y)_{\text{ob}} = \tilde{g}(X, Y) & (b) \quad & g(b_*X, M)_{\text{ob}} = 0 \\ (c) \quad & g(b_*X, N)_{\text{ob}} = 0 & (d) \quad & g(M, N)_{\text{ob}} = 0 \\ (e) \quad & g(M, M)_{\text{ob}} = g(N, N)_{\text{ob}} = 1. \end{aligned}$$

If D is the induced connection on V_{2m-1} , then we have the Gauss equation

$$(2.2) \quad V_{b_*X}b_*Y = b_*D_XY + MH(X, Y) + NK(X, Y)$$

where H and K are symmetric bilinear functions in V_{2m-1} . The Weingarten equations in V_{2m-1} are given by

$$\begin{aligned} V_{b_*X}M &= -b_*'H(X) + l(X)M, & g('H(X), Y) &\stackrel{\text{def}}{=} H(X, Y), \\ V_{b_*X}N &= -b_*'K(X) - l'(X)N, & g('K(X), Y) &\stackrel{\text{def}}{=} K(X, Y). \end{aligned}$$

If the second fundamental forms H and K of V_{2m-1} are of the form $H(X, Y) = \mu_1\tilde{g}(X, Y)$, $K(X, Y) = \mu_2\tilde{g}(X, Y)$ where $\mu_1, \mu_2 = (\text{Tr } b_*)/n'$ then V_{2m-1} is called totally umbilical. In our case, we take $\mu_1 = \mu_2 = \mu$. If the second fundamental form vanishes identically then V_{2m-1} is said to be totally geodesic. (YANO and KON [3]).

A submanifold V_{2m-1} of a Lorentzian Para-contact manifold V_{2m+1} is said to be invariant if the structure vector field ξ of V_{2m-1} is tangent to V_{2m+1} and $\varphi(T_X(V_{2m-1})) \subseteq T_X(V_{2m-1})$ where $T_X(V_{2m-1})$ denotes the tangent space of V_{2m-1} at X . On the other hand if $\varphi(T_X(V_{2m-1})) \subseteq T_X(V_{2m-1})^\perp$ for all $X \in V_{2m-1}$, where $T_X(V_{2m-1})^\perp$ is the normal space of V_{2m-1} at X then V_{2m-1} is said to be antiinvariant in V_{2m+1} .

Let us put

$$(2.3) \quad \begin{aligned} (a) \quad & \varphi(b_*X) = b_*X + \alpha(X)M + \nu(X)N \\ (b) \quad & \xi = b_*\xi + \rho M + \sigma N \\ (c) \quad & \varphi(M) = -b_*p + \delta N \\ (d) \quad & \varphi(N) = -b_*q + \theta M \end{aligned}$$

Pre-multiplying (2.3) (a) by φ and using (1.1), (2.3) (b), (c), (d) we obtain

$$(2.4) \quad \begin{aligned} b_*X + \eta(b_*X)(b_*\xi + \rho M + \sigma N) &= b_*\phi^2X - b_*p\alpha(X) \\ &- b_*q\nu(X) + M(\alpha(\phi(X)) + \theta\nu(X) + N(\nu(\phi(X)) + \delta\alpha(X)) \end{aligned}$$

Substituting from (2.3) (a) in

$$g(\varphi(b_*X), \varphi(b_*Y)) = g(b_*X, b_*Y) + \eta(b_*X)\eta(b_*Y)$$

and using (2.1) we obtain

$$(2.5) \quad \begin{aligned} g(\phi X, \phi Y) &= g(X, Y) + (\eta(b_*X)ob)(\eta(b_*Y)ob) \\ &- \alpha(X)\alpha(Y) - \nu(X)\nu(Y) \end{aligned}$$

Equations (2.4) and (2.5) give

$$\begin{aligned} \varphi^2X &= X + a(X)T \\ g(\varphi X, \varphi Y) &= g(X, Y) + a(X)a(Y) \end{aligned}$$

iff

$$\begin{aligned} (b_*X)ob &= a(X), \quad p(\alpha(X)) + q(\nu(X)) = 0 \\ \rho a(X) &= \alpha(\varphi(X)) + \theta(\nu(X)), \quad \sigma a(X) = \nu(\varphi X) + \delta(\alpha(X)) \\ \alpha(X)\alpha(Y) &+ \nu(X)\nu(Y) = 0. \end{aligned}$$

The above equations are consistent iff

$$(2.6) \quad \eta(b_*X)ob = a(X), \quad \alpha(X) = \nu(X) = 0, \quad \rho = \sigma = 0.$$

Substituting these in (2.3) (a), (b) we obtain

$$(2.7) \quad (a) \quad \varphi(b_*X) = b_*\varphi X \quad (b) \quad \xi = b_*\xi.$$

Thus we have

Theorem 2.1. *The necessary and sufficient conditions for a submanifold of V_{2m+1} to be a Lorentzian Para-contact submanifold are (2.6) and (2.7).*

Theorem 2.2. *Let us denote the Nijenhuis tensors in V_{2m+1} and V_{2m-1} by N and n , determined by φ and ϕ respectively, then $N(b_*X, b_*Y) = b_*n(X, Y)$.*

PROOF. In consequence of (2.2) and (2.7) (a) we have

$$\begin{aligned}\varphi(\varphi[b_*X, b_*Y]) &= \varphi(\varphi(\nabla_{b_*X}b_*Y - \nabla_{b_*Y}b_*X)) \\ &= b_*\phi^2(D_XY) - b_*\phi^2(D_YX) = b_*\phi^2([X, Y])\end{aligned}$$

hence

$$N(b_*X, b_*Y) = b_*n(X, Y).$$

Definition 2.1. An Lorentzian Para-contact manifold is said to be normal if

$$N(X, Y) + d\eta(X, Y)\xi = 0.$$

Theorem 2.3. *If V_{2m+1} is normal then V_{2m-1} is also normal.*

PROOF. We have from Theorem (2.2)

$$N(b_*X, b_*Y) = b_*n(X, Y)$$

$$\begin{aligned}\text{Now } N(b_*X, b_*Y) + ((\nabla_{b_*X}\eta)(b_*Y) - (\nabla_{b_*Y}\eta)(b_*X))\xi &= 0 \\ n(X, Y) + ((D_Xa)(Y) - (D_Ya)(X))\xi &= 0.\end{aligned}$$

This shows that if V_{2m+1} is normal then V_{2m-1} is also normal.

Theorem 2.4. *When V_{2m-1} is an Lorentzian Para-contact submanifold in a Lorentzian Para-contact manifold we have*

$$\begin{aligned}(i) \quad \eta(M) &= 0, & \eta(N) &= 0, \\ (2.8) \quad (ii) \quad \varphi(M) &= \delta N, & \varphi N &= \theta M \\ (iii) \quad \delta &= \theta & \text{and } \delta\theta &= 1.\end{aligned}$$

PROOF. We have $g(\varphi M, b_*X) - g(M, b_*\varphi X) = 0$ which gives

$$g(-b_*p + \delta N, b_*X) - g(M, b_*\varphi X) = 0.$$

Using (2.1) (a), (b), (c), (d) we get

$$g(p, X) = 0 \implies p = 0.$$

Similarly we can get $q = 0$, putting this value in (2.3) (c) and (d) we get

$$\varphi(M) = \delta N, \quad \varphi(N) = \theta M$$

Pre-multiplying (2.3) (c), (d) by φ and using (1.1) and equating tangential and normal parts we have

$$\eta(M) = 0, \quad \eta(N) = 0, \quad \delta\theta = 1.$$

We also have

$$g(\varphi M, N) - g(M, \varphi N) = 0.$$

Putting the values of M and N as above we get

$$\delta = \theta.$$

Theorem 2.5. *Let V_{2m+1} be an Lorentzian Para-contact cosymplectic manifold then V_{2m-1} is also an LP-cosymplectic manifold and*

$$(2.9) \quad H(X, \varphi Y) = \delta K(X, Y), \quad K(X, \varphi Y) = \delta H(X, Y)$$

where H and K are symmetric bilinear functions in V_{2m-1} and $\delta^2 = 1$.

PROOF. We have

$$\begin{aligned} (\nabla_{b_* X} \eta)(b_* Y) = 0 &\implies X(a(Y) - a(D_X Y)) = 0 \\ &\implies (D_X a)(Y) = 0. \end{aligned}$$

Also

$$(\nabla_{b_* X} \varphi)(b_* Y) = 0 \implies \nabla_{b_* X} b_* Y = \varphi(\nabla_{b_* X} b_* Y)$$

which gives

$$\begin{aligned} b_* D_X \varphi Y + H(X, \varphi Y)M + K(X, \varphi Y)N \\ = b_* \varphi(D_X Y) + H(X, Y)\delta N + K(X, Y)\delta M. \end{aligned}$$

This equation implies that $(D_X \varphi)(Y) = 0$ and (2.9) are satisfied. This completes the proof.

Theorem 2.6. *Let V_{2m+1} be an LP-nearly cosymplectic manifold. Then V_{2m-1} is also an LP-nearly cosymplectic manifold and*

$$H(X, \varphi Y) + H(Y, \varphi X) - 2\delta H(X, Y) = 0$$

and

$$K(X, \varphi Y) + K(Y, \varphi X) - 2\delta K(X, Y) = 0$$

where $\delta^2 = 1$.

PROOF. For an LP-nearly cosymplectic manifold we have

$$\begin{aligned} (\nabla_{b_* X} \varphi)(b_* Y) + (\nabla_{b_* Y} \varphi)(b_* X) &= 0 \\ \nabla_{b_* X} b_* \varphi Y + \nabla_{b_* Y} b_* \varphi X &= \varphi(\nabla_{b_* X} b_* Y) + \varphi(\nabla_{b_* Y} b_* X) \end{aligned}$$

or

$$\begin{aligned} b_* D_X \varphi Y + H(X, \varphi Y)M + K(X, \varphi Y)N + b_* D_Y \varphi X \\ + H(Y, \varphi X)M + K(Y, \varphi X)N = b_* \varphi(D_X Y) + b_* \varphi(D_Y X) \\ + H(X, Y)\delta N + H(Y, X)\delta N + K(X, Y)\delta M + K(Y, X)\delta M. \end{aligned}$$

This equation implies

$$(D_X \varphi)(Y) + (D_Y \varphi)(X) = 0$$

and

$$H(X, \varphi Y) + H(Y, \varphi X) - 2\delta K(X, Y) = 0$$

$$K(X, \varphi Y) + K(Y, \varphi X) - 2\delta H(X, Y) = 0.$$

Theorem 2.7. *Let V_{2m-1} be a submanifold tangent to the structure vector field ξ of an Lorentzian Para-Sasakian manifold V_{2m+1} . If V_{2m-1} is totally umbilical then V_{2m-1} is totally geodesic.*

PROOF. From Gauss' equation we have

$$\nabla_{b_*X} \xi = b_*D_X \xi + H(X, \xi)M + K(X, \xi)N,$$

or

$$b_*\varphi X = b_*D_X \xi + H(X, \xi)M + K(X, \xi)N.$$

Equating tangential and normal parts we get

$$\varphi X = D_X \xi \quad \text{and} \quad H(X, \xi) = 0, \quad K(X, \xi) = 0.$$

Thus

$$H(\xi, \xi) = 0, \quad K(\xi, \xi) = 0.$$

If V_{2m-1} is totally umbilical, then $H(X, Y) = \mu g(X, Y) = K(X, Y)$. Writing ξ for both X and Y we get

$$H(\xi, \xi) = K(\xi, \xi) = 0 \implies g(\xi, \xi) = 0 \implies \mu = 0$$

which implies that

$$H(X, Y) = K(X, Y) = 0.$$

Thus V_{2m-1} is totally geodesic.

If V_{2m-1} is totally geodesic then $H(X, \xi) = 0$ that is φX is tangent to V_{2m-1} and hence V_{2m-1} is an invariant submanifold.

Theorem 2.8. *Let V_{2m-1} be a submanifold of a Lorentzian Para-Sasakian manifold. V_{2m+1} is tangent to the structure vector field ξ of V_{2m+1} . Then vector field ξ is parallel with respect to the induced connection on V_{2m-1} if and only if V_{2m-1} is an anti-invariant submanifold in V_{2m+1} .*

PROOF. We have for the tangent ξ of V_{2m-1}

$$(2.10) \quad \nabla_{b_*X} \xi = b_*\varphi X = b_*D_X \xi + H(X, \xi)M + K(X, \xi)N.$$

Since ξ is parallel with respect to the induced connection we have

$$D_X \xi = 0$$

From (2.10) we have

$$\varphi X = H(X, \xi)M + K(X, \xi)N$$

Hence φX is normal to V_{2m-1} . Thus $\varphi X \in T_X(V_{2m-1})^\perp$ for every vector field X on V_{2m-1} . Thus V_{2m-1} is anti-invariant. Conversely if V_{2m-1} is anti-invariant then $\varphi X = H(X, \xi)M + K(X, \xi)N$, hence $D_X \xi = 0$. This completes the proof.

Theorem 2.9. *Let V_{2m-1} be a submanifold of an Lorentzian Para-Sasakian manifold of V_{2m+1} . If the structure vector field ξ is normal to V_{2m-1} , then V_{2m-1} is totally geodesic if and only if V_{2m-1} is an anti-invariant submanifold.*

PROOF. Since ξ is normal to V_{2m-1} we have

$$\begin{aligned} g(b_* \varphi X, b_* Y) &= g(b_* \nabla_X \xi, b_* Y) = g(-b_*' H(X), b_* Y) + g(l(X)\xi, b_* Y) \\ &= g(-b_*' K(X), b_* Y) - g(l'(X)\xi, b_* Y), \end{aligned}$$

or

$$g(b_* \varphi X, b_* Y) = -g('H(X), Y) = g('K(X), Y) \text{ for any } X$$

and Y on V_{2m-1} . Hence Φ , H and K are symmetric, hence $g(b_* \varphi X, b_* Y) = g('H(X), Y) = 0 = g('K(X), Y)$. If V_{2m-1} is totally geodesic then

$$'K(X) = 'H(X) = 0 \implies \varphi(X) \in T_X(V_{2m-1}).$$

Hence V_{2m-1} is anti-invariant.

Conversely if V_{2m-1} is anti-invariant then

$$\begin{aligned} g('H(X), Y) &= 0 = g('K(X), Y) \\ 'H(X) &= 0 = 'K(X) \quad H(X, Y) = 0 = K(X, Y) \end{aligned}$$

hence V_{2m-1} is totally geodesic.

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