

## On certain tensor fields on contact metric manifolds II

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*To the memory of Professor András Rapcsák*

In Sasakian manifolds, MATSUMOTO and CHŪMAN [3] defined a contact Bochner curvature tensor (see also YANO [6]). This is invariant with respect to a  $D$ -homothetic deformation (see TANNO [5] about a  $D$ -homothetic deformation). On the other hand, if a contact metric manifold is normal, it is called a Sasakian manifold. In this paper, we define a new tensor field which is invariant with respect to the  $D$ -homothetic deformation and call it an EH-tensor. Then we shall establish that a contact metric manifold is a Sasakian manifold if and only if its EH-tensor vanishes.

### 1. Preliminaries

Let  $M$  be a  $(2n + 1)$ -dimensional contact metric manifold with the structure tensors  $(\phi, \xi, \eta, g)$ . Then these satisfy

$$\begin{aligned}\phi\xi &= 0, \quad \eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(\xi, X) \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = d\eta(X, Y)\end{aligned}$$

for any vector fields  $X$  and  $Y$  on  $M$ . Now we define an operator  $h$  by  $h = -\frac{1}{2}\mathcal{L}_\xi\phi$ , where  $\mathcal{L}$  denotes Lie differentiation. Then the vector field  $\xi$  is Killing if and only if  $h$  vanishes. It is well known that  $h$  and  $\phi h$  are symmetric operators,  $h$  anti-commutes with  $\phi$  (i.e.,  $\phi h + h\phi = 0$ ),  $h\xi = 0$ ,  $\eta_0 h = 0$ ,  $\text{Tr } h = 0$  and  $\text{Tr } \phi h = 0$ , where  $\text{Tr } h$  denotes the trace of  $h$ . Moreover the following general formulas for a contact metric manifold  $M$  were obtained

$$(1.1) \quad \nabla_X \xi = \phi X + \phi h X$$

$$(1.2) \quad \frac{1}{2}(R(\xi, X)\xi - \phi R(\xi, \phi X)\xi) = h^2 X + \phi^2 X$$

$$(1.3) \quad g(Q\xi, \xi) = 2n - \text{Tr } h^2$$

$$(1.4) \quad \sum_{i=1}^{2n+1} (\nabla_{E_i} \phi) E_i = -2n\xi \quad (\{E_i\} \text{ is an orthonormal frame})$$

$$(1.5) \quad (\nabla_{\phi X} \phi) \phi Y + (\nabla_X \phi) Y = -2g(X, Y)\xi + \eta(Y)(X + hX + \eta(X)\xi) \\ ((\nabla_X \phi)\xi = -\eta(X)\xi + X + hX)$$

$$(1.6) \quad \phi(\nabla_\xi h)X = X - \eta(X)\xi - h^2 X - R(X, \xi)\xi,$$

where  $\nabla$  is the covariant differentiation with respect to  $g$ ,  $Q$  is the Ricci operator of  $M$  and  $R$  is the curvature tensor field of  $M$  (see., [1], [2] and [4]). Moreover, by means of  $\phi h\xi = 0$ , we get

$$(1.7) \quad \phi(\nabla_Y h)\xi = -hY - h^2 Y.$$

If  $\xi$  is Killing in a contact metric manifold  $M$ ,  $M$  is said to be a  $K$ -contact Riemannian manifold. If a contact metric manifold  $M$  is normal (i.e.,  $N + 2d\eta \otimes \xi = 0$ , where  $N$  denotes the Nijenhuis tensor formed with  $\phi$ )  $M$  is called a Sasakian manifold. A Sasakian manifold is a  $K$ -contact Riemannian manifold. In a Sasakian manifold with structure tensors  $(\phi, \xi, \eta, g)$ , we have

$$\nabla_X \xi = \phi X,$$

$$(\nabla_X \phi)Y = R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X, \quad \phi Q = Q\phi, \quad Q\xi = 2n\xi$$

(see e.g., [7]).

## 2. $D$ -homothetic deformations

Let  $M$  be an  $(m+1)$ -dimensional ( $m = 2n$ ) contact metric manifold. Now we define the tensor field  $B^{es}$  in  $M$  by

$$(2.1) \quad B^{es}(X, Y) = R(X, Y) + h\phi X \wedge h\phi Y \\ + \frac{1}{2(m+4)}(QY \wedge X - (\phi Q\phi Y) \wedge X + \frac{1}{2}(\eta(Y)Q\xi \wedge X \\ + \eta(QY)\xi \wedge X) - QX \wedge Y + (\phi Q\phi X) \wedge Y - \frac{1}{2}(\eta(X)Q\xi \wedge Y \\ + \eta(QX)\xi \wedge Y) + (Q\phi Y) \wedge \phi X + (\phi QY) \wedge \phi X \\ - (Q\phi X) \wedge \phi Y - (\phi QX) \wedge \phi Y + 2g(Q\phi X, Y)\phi \\ + 2g(\phi QX, Y)\phi + 2g(\phi X, Y)\phi Q + 2g(\phi X, Y)Q\phi \\ - \eta(X)QY \wedge \xi + \eta(X)(\phi Q\phi Y) \wedge \xi + \eta(Y)QX \wedge \xi \\ - \eta(Y)(\phi Q\phi X) \wedge \xi) - \frac{k+m}{m+4}(\phi Y \wedge \phi X + 2g(\phi X, Y)\phi)$$

$$\begin{aligned}
& -\frac{k-4}{m+4}Y \wedge X + \frac{k}{m+4}(\eta(Y)\xi \wedge X + \eta(X)Y \wedge \xi) \\
& -\frac{1}{(m+4)(m+2)}\text{Tr } h^2(\phi Y \wedge \phi X + 2g(\phi X, Y)\phi \\
& + Y \wedge X + \eta(X)\xi \wedge Y - \eta(Y)\xi \wedge X),
\end{aligned}$$

where  $k = \frac{S+m}{m+2}$  ( $S$  is the scalar curvature tensor of  $M$ ) and  $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$  (cf., [3]). Using (2.1),  $B^{es}$  satisfies the following identities in a contact metric manifold  $M$ :

$$\begin{aligned}
(2.2) \quad & B^{es}(X, Y)Z = -B^{es}(Y, X)Z \\
& B^{es}(X, Y)Z + B^{es}(Y, Z)X + B^{es}(Z, X)Y = 0 \\
& g(B^{es}(X, Y)Z, W) = -g(Z, B^{es}(X, Y)W), \\
& g(B^{es}(X, Y)Z, W) = g(B^{es}(Z, W)X, Y).
\end{aligned}$$

If  $M$  is a Sasakian manifold,  $B^{es}$  coincides with the contact Bochner curvature tensor by MASTUMOTO and CHŪMAN [3] and the following are satisfied:

$$\begin{aligned}
(2.3) \quad & B^{es}(\xi, Y)Z = B^{es}(X, Y)\xi = 0 \quad \text{and} \\
& B^{es}(\phi X, \phi Y)Z = B^{es}(X, Y)Z,
\end{aligned}$$

where we have used  $R(\phi X, Y)Z - R(\phi Y, X)Z = g(\phi Z, X)Y - g(\phi Z, Y)X - g(Z, X)\phi Y + g(Z, Y)\phi X$  in a Sasakian manifold.

We consider a  $D$ -homothetic deformation  $g^* = \alpha g + \alpha(\alpha-1)\eta \otimes \eta$ ,  $\phi^* = \phi$ ,  $\xi^* = \alpha^{-1}\xi$ ,  $\eta^* = \alpha\eta$  on a contact metric manifold  $M$ , where  $\alpha$  is a positive constant. For a  $D$ -homothetic deformation we say that  $M(\phi, \xi, \eta, g)$  is  $D$ -homothetic to  $M(\phi^*, \xi^*, \eta^*, g^*)$ . It is well known that if a contact metric manifold  $M(\phi, \xi, \eta, g)$  is  $D$ -homothetic to  $M(\phi^*, \xi^*, \eta^*, g^*)$ , then  $M(\phi^*, \xi^*, \eta^*, g^*)$  is a contact metric manifold. Moreover if  $M(\phi, \xi, \eta, g)$  is a  $K$ -contact Riemannian manifold (resp. Sasakian manifold), then  $M(\phi^*, \xi^*, \eta^*, g^*)$  is also a  $K$ -contact Riemannian manifold (resp. Sasakian manifold) (see [5]). Denoting by  $W^i_{jk}$  the difference  ${}^*\Gamma^i_{jk} - \Gamma^i_{jk}$  of Christoffel symbols, by (1.1) we have in a contact metric manifold  $M$

$$\begin{aligned}
W(X, Y) &= (\alpha-1)(\eta(Y)\phi X + \eta(X)\phi Y) \\
&\quad + \frac{\alpha-1}{2\alpha} ((\nabla_X \eta)(Y) + (\nabla_Y \eta)(X))\xi \quad (\text{see [5]}) \\
&= (\alpha-1)(\eta(Y)\phi X + \eta(X)\phi Y) + \frac{\alpha-1}{\alpha} g(\phi X, Y)\xi.
\end{aligned}$$

Putting this into

$$R^*(X, Y)Z = R(X, Y)Z + (\nabla_X W)(Z, Y) - (\nabla_Y W)(Z, X) \\ + W(W(Z, Y), X) - W(W(Z, X), Y)$$

and using (1.1) we have

$$(2.4) \quad R^*(X, Y)Z = R(X, Y)Z + (\alpha - 1)(2g(\phi X, Y)\phi Z \\ + g(\phi Z, Y)\phi X - g(\phi Z, X)\phi Y \\ + \eta(Y)(\nabla_X \phi)(Z) + \eta(Z)(\nabla_X \phi)(Y) \\ - \eta(X)(\nabla_Y \phi)(Z) - \eta(Z)(\nabla_Y \phi)(X)) - (\alpha - 1)^2(\eta(Z)\eta(X)Y \\ - \eta(Z)\eta(Y)X) - \frac{\alpha - 1}{\alpha}(g(X, (\nabla_Y \phi)hZ)\xi - g(Y, (\nabla_X \phi)hZ)\xi \\ + g(X, \phi(\nabla_Y h)Z)\xi - g(Y, \phi(\nabla_X h)Z)\xi \\ + g(X, \phi hZ)\phi hY - g(Y, \phi hZ)\phi hX) \\ - \frac{(\alpha - 1)^2}{\alpha}(\eta(X)g(hZ, Y)\xi - \eta(Y)g(hZ, X)\xi).$$

Choosing a  $\phi^*$ -basis with respect to  $g^*$  (if we take a  $\phi^*$ -basis  $(\xi^*, X_1, \phi^* X_1, \dots, X_n, \phi^* X_n)$ , a  $\phi$ -basis with respect to  $g$  is  $(\xi, \sqrt{\alpha}X_1, \sqrt{\alpha}\phi X_1, \dots, \sqrt{\alpha}X_n, \sqrt{\alpha}\phi X_n)$ ) and using (1.4) and (1.6), we get

$$(2.5) \quad \text{Ric}^*(X, Y) = \text{Ric}(X, Y) + (\alpha - 1)(-2g(X, Y) \\ + 2(2n + 1)\eta(X)\eta(Y)) + 2n(\alpha - 1)^2\eta(X)\eta(Y) \\ - \frac{\alpha - 1}{\alpha}(-g(X, Y) + \eta(X)\eta(Y) - 2g(hX, Y) \\ + g(hX, hY) + g(R(X, \xi)\xi, Y)),$$

where  $\text{Ric}$  is the Ricci curvature of  $M$ . From (2.5), we find

$$(2.6) \quad Q^*X = \frac{1}{\alpha}QX + \frac{\alpha - 1}{\alpha}(-2X + 2(2n + 1)\eta(X)\xi) \\ - \frac{\alpha - 1}{\alpha^2}g(X, Q\xi)\xi - 2n\left(\frac{\alpha - 1}{\alpha}\right)^2\eta(X)\xi \\ - \frac{\alpha - 1}{\alpha^2}(-X + \eta(X)\xi - 2hX + h^2X + R(X, \xi)\xi),$$

where we have used  $Q^*\xi = \frac{1}{\alpha}Q\xi - \frac{\alpha-1}{\alpha^2}g(Q\xi, \xi)\xi + 2n\frac{\alpha^2-1}{\alpha^2}\xi$ .

By virtue of (1.3) we have

$$(2.7) \quad S^* = \frac{1}{\alpha}S - 2n\frac{\alpha - 1}{\alpha} + \frac{\alpha - 1}{\alpha^2}\text{Tr } h^2.$$

Moreover, using the definition of  $h$ , we have

$$(2.8) \quad h^* = \frac{1}{\alpha} h,$$

from which we get

$$(2.9) \quad \text{Tr } h^{*2} = \frac{1}{\alpha^2} \text{Tr } h^2.$$

By means of (1.2), (2.1), (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9), after some lengthy computation, we obtain

$$(2.10) \quad \begin{aligned} {}^*B^{es}(X, Y)Z &= B^{es}(X, Y)Z + (\alpha - 1)(\eta(Y)g(X, Z)\xi \\ &\quad - \eta(X)g(Y, Z)\xi + 2\eta(X)\eta(Z)Y - 2\eta(Y)\eta(Z)X \\ &\quad + \eta(Y)(\nabla_X \phi)Z + \eta(Z)(\nabla_X \phi)Y - \eta(X)(\nabla_Y \phi)Z \\ &\quad - \eta(Z)(\nabla_Y \phi)X) - \frac{\alpha - 1}{\alpha}(g(X, (\nabla_Y \phi)hZ)\xi \\ &\quad - g(Y, (\nabla_X \phi)hZ)\xi + g(X, \phi(\nabla_Y h)Z)\xi - g(Y, \phi(\nabla_X h)Z)\xi) \\ &\quad - \frac{(\alpha - 1)^2}{\alpha}(\eta(X)g(hZ, Y)\xi - \eta(Y)g(hZ, X)\xi) \\ &\quad + \frac{1}{2(2n + 4)} \left\{ \frac{3(\alpha - 1)}{2\alpha}(\eta(X)\eta(Z)g(Y, Q\xi)\xi \right. \\ &\quad - \eta(Y)\eta(Z)g(X, Q\xi)\xi + g(Y, Z)g(X, Q\xi)\xi \\ &\quad - g(X, Z)g(Y, Q\xi)\xi) + \frac{1}{2} \frac{\alpha - 1}{\alpha}(g(Y, Z)\eta(X) \\ &\quad - g(X, Z)\eta(Y))g(Q\xi, \xi)\xi \\ &\quad + \frac{\alpha - 1}{\alpha}(-g(\phi X, Z)g(\phi Y, Q\xi)\xi + g(\phi Y, Z)g(\phi X, Q\xi)\xi \\ &\quad - 2g(\phi X, Y)g(\phi Z, Q\xi)\xi \\ &\quad \left. + 4ng(X, Z)\eta(Y)\xi - 4ng(Y, Z)\eta(X)\xi) \right\}. \end{aligned}$$

Now we introduce in  $M$  an  $EH$ -tensor by

$$(2.11) \quad \begin{aligned} EH(X, Y)Z &= B^{es}(X, Y)Z - \eta(X)B^{es}(\xi, Y)Z \\ &\quad - \eta(Y)B^{es}(X, \xi)Z - \eta(Z)B^{es}(X, Y)\xi \\ &\quad - \eta(B^{es}(X, Y)Z)\xi - B^{es}(\phi X, \phi Y)Z \\ &\quad + \eta(Z)B^{es}(\phi X, \phi Y)\xi + \eta(B^{es}(\phi X, \phi Y)Z)\xi \\ &\quad + \eta(X)\eta(B^{es}(\xi, Y)Z)\xi + \eta(Y)\eta(B^{es}(X, \xi)Z)\xi \end{aligned}$$

$$+ \eta(Y)\eta(Z)\phi B^{es}(\phi X, \xi)\xi + \eta(X)\eta(Z)\phi B^{es}(\xi, \phi Y)\xi.$$

In particular, if  $M$  is Sasakian, then  $EH = 0$  from (2.2) and (2.3).

**Theorem 2.1.** *The  $EH$ -tensor is invariant with respect to the  $D$ -homothetic deformation  $M(\phi, \xi, \eta, g) \rightarrow M(\phi^*, \xi^*, \eta^*, g^*)$  on a contact metric manifold  $M$ .*

PROOF. Using (1.5), (1.6), (1.7) and (2.10), we find

$$\begin{aligned}
 (2.12) \quad & -\eta^*(X)\overset{*}{B}^{es}(\xi^*, Y)Z = -\eta(X)\overset{*}{B}^{es}(\xi, Y)Z \\
 & = -\eta(X)B^{es}(\xi, Y)Z + (\alpha - 1)(-\eta(X)\eta(Z)Y \\
 & + \eta(X)\eta(Z)hY + \eta(X)(\nabla_Y \phi)Z) \\
 & + \frac{\alpha - 1}{\alpha}(\eta(X)\eta(Y)\eta(Z)\xi + \eta(X)g(Y, R(Z, \xi)\xi)\xi) \\
 & + \frac{(\alpha - 1)^2}{\alpha}\eta(X)g(Y, Z)\xi + \frac{(\alpha - 1)(\alpha - 2)}{\alpha}\eta(X)g(Y, hZ)\xi \\
 & + \frac{1}{2(2n + 4)}\left\{ \frac{3(\alpha - 1)}{2\alpha}(-g(Y, Z)\eta(X)g(Q\xi, \xi)\xi \right. \\
 & + \eta(X)\eta(Y)\eta(Z)g(Q\xi, \xi)\xi) \\
 & + \frac{1}{2}\frac{\alpha - 1}{\alpha}(-\eta(X)g(Y, Z)g(Q\xi, \xi)\xi \\
 & + \eta(X)\eta(Y)\eta(Z)g(Q\xi, \xi)\xi) \\
 & \left. + \frac{\alpha - 1}{\alpha}(4ng(Y, Z)\eta(X)\xi - 4n\eta(X)\eta(Y)\eta(Z)\xi) \right\}
 \end{aligned}$$

$$\begin{aligned}
 (2.13) \quad & -\eta^*(Y)\overset{*}{B}^{es}(X, \xi^*)Z \\
 & = -\eta(Y)B^{es}(X, \xi)Z + (\alpha - 1)(\eta(Y)\eta(Z)X \\
 & - \eta(Y)\eta(Z)hX - \eta(Y)(\nabla_X \phi)Z) - \frac{\alpha - 1}{\alpha}(\eta(X)\eta(Y)\eta(Z)\xi \\
 & + \eta(Y)g(X, R(Z, \xi)\xi)\xi) - \frac{(\alpha - 1)^2}{\alpha}g(X, Z)\eta(Y)\xi \\
 & - \frac{(\alpha - 1)(\alpha - 2)}{\alpha}g(hX, Z)\eta(Y)\xi \\
 & + \frac{1}{2(2n + 4)}\left\{ \frac{3(\alpha - 1)}{2\alpha}(g(X, Z)\eta(Y)g(Q\xi, \xi)\xi \right. \\
 & \left. - \eta(X)\eta(Y)\eta(Z)g(Q\xi, \xi)\xi) \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \frac{\alpha - 1}{\alpha} (g(X, Z)\eta(Y)g(Q\xi, \xi)\xi \\
& - \eta(X)\eta(Y)\eta(Z)g(Q\xi, \xi)\xi) \\
& + \frac{\alpha - 1}{\alpha} (-4ng(X, Z)\eta(Y)\xi + 4n\eta(X)\eta(Y)\eta(Z)\xi) \Big\}
\end{aligned}$$

$$\begin{aligned}
(2.14) \quad & -\eta^*(Z)\overset{*}{B}^{es}(X, Y)\xi^* = -\eta(Z)B^{es}(X, Y)\xi \\
& + (\alpha - 1)(-\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X - \eta(Y)\eta(Z)hX \\
& + \eta(X)\eta(Z)hY - \eta(Z)(\nabla_X \phi)Y + \eta(Z)(\nabla_Y \phi)X)
\end{aligned}$$

$$\begin{aligned}
(2.15) \quad & -\eta^*(\overset{*}{B}^{es}(X, Y)Z)\xi^* = -\eta(B^{es}(X, Y)Z)\xi \\
& - \frac{\alpha - 1}{\alpha} (\eta(X)g(hY, Z)\xi - \eta(Y)g(hX, Z)\xi \\
& - g(X, (\nabla_Y \phi)hZ)\xi + g(Y, (\nabla_X \phi)hZ)\xi - g(X, \phi(\nabla_Y h)Z)\xi \\
& + g(Y, \phi(\nabla_X h)Z)\xi) + \frac{1}{2(2n+4)} \left\{ \frac{3(\alpha - 1)}{2\alpha} \right. \\
& (-g(Y, Z)g(X, Q\xi)\xi + g(X, Z)g(Y, Q\xi)\xi \\
& - \eta(X)\eta(Z)g(Y, Q\xi)\xi + \eta(Y)\eta(Z)g(X, Q\xi)\xi) \\
& + \frac{1}{2} \frac{\alpha - 1}{\alpha} (-\eta(X)g(Y, Z)g(Q\xi, \xi)\xi \\
& + \eta(Y)g(X, Z)g(Q\xi, \xi)\xi) + \frac{\alpha - 1}{\alpha} (g(\phi X, Z)g(\phi Y, Q\xi)\xi \\
& - g(\phi Y, Z)g(\phi X, Q\xi)\xi + 2g(\phi X, Y)g(\phi Z, Q\xi)\xi \\
& \left. - 4ng(X, Z)\eta(Y)\xi + 4ng(Y, Z)\eta(X)\xi) \right\}.
\end{aligned}$$

Using (1.3), (1.5), (1.6) and (2.15), we get

$$\begin{aligned}
(2.16) \quad & \eta^*(X)\eta^*(\overset{*}{B}^{es}(\xi^*, Y)Z)\xi^* + \eta^*(Y)\eta^*(\overset{*}{B}^{es}(X, \xi^*)Z)\xi^* \\
& = \frac{\alpha - 1}{\alpha} (2g(Y, hZ)\eta(X)\xi - 2g(X, hZ)\eta(Y)\xi \\
& + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\
& - g(Y, R(Z, \xi)\xi)\eta(X)\xi + g(X, R(Z, \xi)\xi)\eta(Y)\xi) \\
& + \frac{1}{2n+4} \frac{\alpha - 1}{\alpha} (g(X, Z)\eta(Y) - g(Y, Z)\eta(X)) \operatorname{Tr} h^2
\end{aligned}$$

From (1.5) and (2.14) we have

$$\begin{aligned}
 (2.17) \quad & \eta^*(Y)\eta^*(Z)\phi^*B^{es}(\phi^*X, \xi^*)\xi^* \\
 &= \eta(Y)\eta(Z)\phi B^{es}(\phi X, \xi)\xi + 2(\alpha - 1)\eta(Y)\eta(Z)hX \\
 & \eta^*(X)\eta^*(Z)\phi^*B^{es}(\xi^*, \phi^*Y)\xi^* \\
 &= -\eta(X)\eta(Z)\phi B^{es}(\xi, \phi Y)\xi - 2(\alpha - 1)\eta(X)\eta(Z)hY.
 \end{aligned}$$

Therefore, using (1.3), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and (2.17), we get

$$\begin{aligned}
 (2.18) \quad & B^{es}(X, Y)Z - \eta^*(X)B^{es}(\xi^*, Y)Z \\
 & - \eta^*(Y)B^{es}(X, \xi^*)Z - \eta^*(Z)B^{es}(X, Y)\xi^* \\
 & - \eta^*(B^{es}(X, Y)Z)\xi^* + \eta^*(X)\eta^*(B^{es}(\xi^*, Y)Z)\xi^* \\
 & + \eta^*(Y)\eta^*(B^{es}(X, \xi^*)Z)\xi^* + \eta^*(Y)\eta^*(Z)\phi^*B^{es}(\phi^*X, \xi^*)\xi^* \\
 & + \eta^*(X)\eta^*(Z)\phi^*B^{es}(\xi^*, \phi^*Y)\xi^* \\
 &= B^{es}(X, Y)Z - \eta(X)B^{es}(\xi, Y)Z - \eta(Y)B^{es}(X, \xi)Z \\
 & - \eta(Z)B^{es}(X, Y)\xi - \eta(B^{es}(X, Y)Z)\xi \\
 & + \eta(X)\eta(B^{es}(\xi, Y)Z)\xi + \eta(Y)\eta(B^{es}(X, \xi)Z)\xi \\
 & + \eta(Y)\eta(Z)\phi B^{es}(\phi X, \xi)\xi + \eta(X)\eta(Z)\phi B^{es}(\xi, \phi Y)\xi.
 \end{aligned}$$

Here, substituting  $\phi^*X$  for  $X$  and  $\phi^*Y$  for  $Y$  in (2.18), we obtain our result.

### 3. Contact metric manifolds with vanishing $EH$ -tensor

We define

$$(3.1) \quad s^\# = \sum_{i,j=1}^{2n+1} g(R(E_i, E_j)\phi E_j, \phi E_i),$$

where  $\{E_i\}$  is an orthonormal frame.

**Lemma 3.1** ([4]). *For any  $(2n + 1)$ -dimensional contact metric manifold  $M$ , we have*

$$s^\# - S + 4n^2 = \text{Tr } h^2 + \frac{1}{2}\{\|\nabla\phi\|^2 - 4n\} \geq 0.$$



Moreover  $M$  is Sasakian if and only if  $\|\nabla\phi\|^2 - 4n = 0$  or equivalently

$$s^\# - S + 4n^2 = 0.$$

**Theorem 3.1.** *Let  $M$  be a contact metric manifold.  $M$  is a Sasakian manifold if and only if the  $EH$ -tensor vanishes.*

PROOF. Since the  $EH$ -tensor of  $M$  vanishes, we have

$$\begin{aligned} (3.2) \quad & g(B^{es}(X, Y)Z, W) - g(B^{es}(\phi X, \phi Y)Z, W) \\ &= \eta(X)g(B^{es}(\xi, Y)Z, W) + \eta(Y)g(B^{es}(X, \xi)Z, W) \\ &+ \eta(Z)g(B^{es}(X, Y)\xi, W) + \eta(W)\eta(B^{es}(X, Y)Z) \\ &- \eta(Z)g(B^{es}(\phi X, \phi Y)\xi, W) - \eta(W)\eta(B^{es}(\phi X, \phi Y)Z) \\ &- \eta(X)\eta(W)\eta(B^{es}(\xi, Y)Z) - \eta(Y)\eta(W)\eta(B^{es}(X, \xi)Z) \\ &- \eta(Y)\eta(Z)g(\phi B^{es}(\phi X, \xi)\xi, W) \\ &- \eta(X)\eta(Z)g(\phi B^{es}(\xi, \phi Y)\xi, W). \end{aligned}$$

Taking  $X = E_i$ ,  $Y = E_j$ ,  $Z = \phi E_j$ ,  $W = \phi E_i$  ( $\{E_i\}$  is a  $\phi$ -basis) in each member of (3.2) and summing over  $i$  and  $j$ , we obtain

$$\begin{aligned} & \sum_{i,j=1}^{2n+1} g(B^{es}(E_i, E_j)\phi E_j, \phi E_i) = s^\# - S + 4n^2 - 2 \operatorname{Tr} h^2 + (\operatorname{Tr} h^2)^2 \\ & \sum_{i,j=1}^{2n+1} g(B^{es}(\phi E_i, \phi E_j)\phi E_j, \phi E_i) = S - 2g(Q\xi, \xi) + \frac{2(n+1)}{n+2}(g(Q\xi, \xi) - S) \\ & + \frac{3n(k+2n)}{n+2} + \frac{n(2n-1)(k-4)}{n+2} - \frac{2}{n+2} \operatorname{Tr} h^2. \end{aligned}$$

Thus, from (1.3) and  $k = \frac{S+2n}{2n+2}$ , we get

$$s^\# - S + 4n^2 - 2 \operatorname{Tr} h^2 + (\operatorname{Tr} h^2)^2 = 0.$$

On the other hand, from (3.2) we find

$$\sum_{i=1}^{2n+1} g(B^{es}(E_i, \xi)\xi, E_i) = - \sum_{i=1}^{2n+1} g(B^{es}(\phi E_i, \xi)\xi, \phi E_i).$$

Hence we have  $g(Q\xi, \xi) = 2n$ .

By Lemma 3.1 we have our result.

### References

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