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# The residual nilpotency of the augmentation ideal

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## 1. Introduction

Let $R$ be a commutative ring with unity, $G$ a group and $RG$ its group ring and let $A(RG)$ denote the *augmentation ideal* of $RG$, that is the kernel of the ring homomorphism $RG \to R$ which maps the group elements to 1. It is easy to see that an $R$-module $A(RG)$ is a free module with the elements $g-1$ $(1 \neq g \in G)$ as a basis.

The ideal $A(RG)$ of the group ring $RG$ is said to be *residually nilpotent* if $\bigcap\limits_{n=1}^{\infty} A^n(RG) = 0$. For convenience, we adopt the following notation: $A^\omega(RG) = \bigcap\limits_{n=1}^{\infty} A^n(RG)$.

We are interested in the residual nilpotence of the augmentation ideals of group rings.

In the case when $R$ is a field, the question about the residual nilpotence of the augmentation ideal is completely solved (see in particular [6], VI, Theorem 2.26).

If $R$ is the ring of integers and if the finitely generated group $G$ has torsion elements then the question about the residual nilpotency of the augmentation ideal is solved in [1].

In [5] the author gives a complete characterization of the residual nilpotence of the augmentation ideal for a group ring over the integers.

For a group ring $RG$ Hartley (see [3], Theorem E) gives a sufficient condition for the residual nilpotence of the augmentation ideal in the case when $\bigcap\limits_{n=1}^{\infty} p^n R = 0$ and the group $G$ has a finite $N$ series.

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In this paper we give sufficient conditions for the residual nilpotence of the augmentation ideal for all arbitrary group ring $RG$ (Theorem 3.1). These conditions are also necessary if the group $G$ has a generalized torsion element (Theorem 3.2). In the case when $G$ is without a generalized torsion element and the torsion group of the additive group of a ring $R$ for some prime $p$ has no elements of infinite $p$-height, the question about the residual nilpotence of the augmentation ideal is solved in Theorem 3.3.

## 2. Notations and some known facts

If $H$ is a normal subgroup of $G$, then $I(RH)$ (or $I(H)$ for short when it is obvious from the context what ring $R$ we are working with) denotes the ideal of $RG$ generated by all elements of the form $h-1$, $h \in H$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\psi^* : RG \to RG/H$ induced by the group homomorphism $\psi$ of $G$ onto $G/H$.

If $K$, $L$ are two subgroups of $G$, then $[K, L]$ denotes the subgroup generated by all commutators $[g,h] = g^{-1}h^{-1}gh$, $g \in K$, $h \in L$.

Let $p$ be a prime and $n$ a natural number. Then $G^{p^n}$ is the subgroup generated by all elements of the form $g^{p^n}$, $g \in G$.

The subgroup $W_p(G)$ defined by $W_p(G) = \bigcap\limits_{n=1}^{\infty} G^{p^n}\gamma_n(G)$, where $\gamma_n(G)$ is the $n$-th term of the lower central series of $G$, i.e. $\gamma_1(G) = G$, $\gamma_2(G) = G'$ is the commutator subgroup $[G,G]$ of $G$, and $\gamma_n(G) = [\gamma_{n-1}(G), G]$.

The ideal $J_p(R)$ of a ring $R$ is defined by $J_p(R) = \bigcap\limits_{n=1}^{\infty} p^n R$.

If $\mathfrak{C}$ denotes a class of groups (by which we understand that $\mathfrak{C}$ contains all groups of order 1 and, with each $H \in \mathfrak{C}$, all isomorphic copies of $H$), we define the class $\mathrm{R}\mathfrak{C}$ of *residually*-$\mathfrak{C}$ groups by letting $G \in \mathrm{R}\mathfrak{C}$ if and only if: whenever $1 \neq g \in G$, there exists a normal subgroup $H_g$ of the group $G$ such that $G/H_g \in \mathfrak{C}$ and $g \notin H_g$.

We use the following notations for standard group classes: $\mathfrak{N}_0$ – torsion-free nilpotent groups, $\mathfrak{N}_p$ – nilpotent $p$-groups of finite exponent, that is, nilpotent groups in which for some $n = n(G)$ every element $g$ satisfies the equation $g^{p^n} = 1$.

Let $\mathfrak{C}$ be a class of groups. A group $G$ is said to be *discriminated* by $\mathfrak{C}$ if for every finite subset $g_1, g_2, \ldots, g_n$ of distinct elements of $G$, there exists a group $H \in \mathfrak{C}$ and a homomorphism $\phi$ of $G$ into $H$, such that $\phi(g_i) \neq \phi(g_j)$ for $g_i \neq g_j$, $(1 \leq i, j \leq n)$.

A series $G = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq \cdots$ of normal subgroups of a group $G$ is called an *N-series* if $[H_i, H_j] \subseteq H_{i+j}$ for all $i, j \geq 1$ and also each of the Abelian groups $H_i/H_j$ is a direct product of (possibly infinitely many) cyclic group which are either infinite or of order $p^k$, where $p$ is a fixed prime and $k$ bounded by some integer depending only on $G$.

It is easy to see that the lower central series of a nilpotent $p$-group of finite exponent is an $N$-series.

The *$n$-th dimension subgroup* $D_n(RG)$ of $G$ over $R$ is the set of group elements $g \in G$ such that $g-1$ lies in the $n$-th power of $A(RG)$. It is well known that for every natural number $n$ the inclusion

$$\gamma_n(G) \subseteq D_n(RG)$$

holds.

An element $g$ of a group $G$ is called a *generalized torsion element* if for all natural numbers $n$ the order of the element $g\gamma_n(G)$ of the factor group $G/\gamma_n(G)$ is finite.

It is clear that the torsion elements of the group $G$ are generalized torsion elements of $G$.

If $g \in G$ is a generalized torsion element then $\Omega_g$ denotes the set of prime divisors of the orders of the elements $g\gamma_n(G) \in G/\gamma_n(G)$ for all $n = 2, 3, \dots$.

**Lemma 2.1.** *For every natural number $n$ the inclusions*

$$I(\gamma_n(G)) \subseteq I(D_n(RG)) \subseteq A^n(RG)$$

*hold.*

The statement is well known.

**Lemma 2.2.** *Let $G$ be discriminated by a class of groups $\mathfrak{C}$ and let $x$ be a nonzero element of $RG$. Then there exists a group $H \in \mathfrak{C}$ and a homomorphism $\phi$ of $RG$ into $RH$ such that $\phi(x) \neq 0$.*

The proof is evident.

**Lemma 2.3.** *If $G$ is discriminated by a class of groups $\mathfrak{C}$ and for each $H \in \mathfrak{C}$ the equality $A^\omega(RH) = 0$ holds, then $A^\omega(RG) = 0$.*

The lemma follows immediately from Lemma 2.2.

**Lemma 2.4.** *Let a class $\mathfrak{C}$ of groups be closed for the taking of subgroups (that is all subgroups of any member of the class $\mathfrak{C}$ are again in the class $\mathfrak{C}$) and also for finite direct products and let $G$ be a residually-$\mathfrak{C}$ group. Then $G$ is discriminated by $\mathfrak{C}$.*

The proof can be obtained immediately.

In this paper we shall use the following theorems:

**Theorem 2.1** ([3], Theorem E)**.** *Let $G$ be a group with a finite $N$-series and $R$ be a commutative ring with unity satisfying $J_p(R) = 0$. Then $A^\omega(RG) = 0$.*

**Theorem 2.2.** ([6], VI, Theorem 2.15). *If $G$ is a residually torsion-free nilpotent group and $R$ is a commutative ring with unity such that its additive group is torsion-free, then $A^{\omega}(RG)=0$.*

## 3. The residual nilpotency of the augmentation ideal

In this section $R$ is a commutative ring with unity.

**Lemma 3.1.** *Let $g\in G$ and $g^{p^n}\in D_t(RG)$ for a prime $p$ and a natural number $n$. Then there exists a natural number $m$ such that*

$$p^m(g-1)\in A^t(RG).$$

Proof. We prove this by induction on $t$. For $t=1$ the statement is obvious. Let $p^s(g-1)\in A^{t-1}(RG)$ for some $s$. From the decomposition $g^{p^n}$ as $(g-1+1)^{p^n}$ we have that

$$g^{p^m}-1=p^m(g-1)+\sum_{i=2}^{t-1}\binom{p^m}{i}(g-1)^i+\sum_{i=t}^{p^m}\binom{p^m}{i}(g-1)^i$$

for every $m$. If $m\geq n(s+t)$, then $p^s$ divides $\binom{p^m}{i}$ ($\binom{p^m}{i}$ is the binomial coefficient $p^m$ over $i$) for $i=1,2,\dots,t-1$ and $g^{p^m}\in D_t(RG)$. Therefore we have

$$g^{p^m}-1=p^m(g-1)+p^s(g-1)^2\sum_{i=2}^{t-1}d_i(g-1)^{i-2}+\sum_{i=t}^{p^m}\binom{p^m}{i}(g-1)^i$$

where $d_ip^s=\binom{p^m}{i}$ for $i=2,3,\dots,t-1$. By Lemma 2.1 $g^{p^m}-1\in A^t(RG)$. Then from the induction hypothesis and from the preceding identity $p^m(g-1)\in A^t(RG)$ follows. $\square$

**Lemma 3.2.** *Let $h\in G^{p^n}\gamma_n(G)$ for a natural number $n$. Then for all natural numbers $t$ and $s$ for which $n\geq t+s$*

$$h-1\equiv p^sF_t(h)\pmod{A^t(RG)}$$

*holds, where $F_t(h)\in A(RG)$.*

Proof. Writing the element $h$ as $h=h_1^{p^n}h_2^{p^n}\dots h_m^{p^n}y_n$ ($h_i\in G$, $y_n\in\gamma_n(G)$) and using the identity

$$xy-1=(x-1)(y-1)+(x-1)+(y-1)\tag{1}$$

we have

$$h-1=\left(h_1^{p^n}h_2^{p^n}\cdots h_m^{p^n}-1\right)(y_n-1)+\left(h_1^{p^n}h_2^{p^n}\cdots h_m^{p^n}-1\right)+(y_n-1).$$

Since $t\le n$, by Lemma 2.1, $(y_n-1)\in A^t(RG)$. It is clear that $p^s$ divides $\binom{p^n}{j}$ for $j=1,2,\dots,t-1$. Then from the preceding identity

$$h-1\equiv\sum_{i=1}^{m}\left(h_i^{p^n}-1\right)b_i\equiv p^s\sum_{i=1}^{m}\sum_{j=1}^{t-1}d_j\,(h_i-1)^j\,b_i\equiv$$
$$\equiv p^sF_t(h)\pmod{A^t(RG)}$$

follows, where $F_t(h)=\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{t-1}d_j\,(h_i-1)^j\,b_i$, $b_i\in RG$ and $p^sd_j=\binom{p^n}{j}$.

□

We recall that if $g$ is a generalized torsion element of a group $G$ then $\Omega_g$ is the set of the prime divisors of the orders of the elements $g\gamma_kG\in G/\gamma_k(G)$ for all $k=2,3,\dots$ and also that $J_p(R)=\bigcap\limits_{n=1}^{\infty}p^nR$.

**Lemma 3.3.** *Let $g$ be a generalized torsion element of a group $G$, $\Lambda$ an arbitrary subset of $\Omega_g$, $r\in\bigcap\limits_{p\in\Lambda}J_p(R)$ and let $x\in\bigcap\limits_{p\in\Omega_g\setminus\Lambda}\bigcap\limits_{n=1}^{\infty}I\left(G^{p^n}\gamma_n(G)\right)$. Then one of the following statement holds:*

1) *if $\Lambda$ is the proper subset of $\Omega_g$, then $r(g-1)x\in A^\omega(RG)$;*
2) *if $\Lambda=\Omega_g$, then $r(g-1)\in A^\omega(RG)$;*
3) *if $\Lambda=\emptyset$, then $(g-1)x\in A^\omega(RG)$.*

Proof. Clearly, it is enough to show that for an arbitrary natural number $t$ the elements $r(g-1)$, $(g-1)x$, $r(g-1)x$ all lie in the ideal $A^t(RG)$.

If $g\in\gamma_t(G)$ then, by Lemma 2.1, $(g-1)\in A^t(RG)$ and the statements follow.

Now let $g\notin\gamma_t(G)$ and let $n_t=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the order of the element $g\gamma_t(G)$ of $G/\gamma_t(G)$. We can renumber the primes so that $p_1,p_2,\dots,p_k\in\Lambda$ and $p_i\notin\Lambda$ for all $i>k$.

Let $g\gamma_t(G)=g_1g_2\cdots g_s\gamma_t(G)$ be a decomposition of the element $g\gamma_t(G)$ of the nilpotent group $G/\gamma_t(G)$ into a product of $p_i$-elements $g_i\gamma_t(G)$ such that $g_i^{q_i}\in\gamma_t(G)$, where $q_i{=}p_i^{\alpha_i}$, $i=1,2,\dots,s$. Then $g=g_1g_2\cdots g_sy_t$ for suitable $y_t\in\gamma_t(G)$. From (1) we conclude that

$$g-1=v+w+(y_t-1)$$

where $v = \sum\limits_{i=1}^{k} (g_i - 1)a_i$, $w = \sum\limits_{i=k+1}^{s} (g_i - 1)a_i$ and $a_i \in RG$.

(If $\Lambda \cap \{p_1, p_2, \ldots, p_s\} = \emptyset$ we assume that $v = 0$, $k = 0$, and in the case $\Lambda \cap \{p_1, p_2, \ldots, p_s\} = \{p_1, p_2, \ldots, p_s\}$ we assume that $w = 0$ and $k = s$.) Since $y_t - 1 \in A^t(RG)$,

$$g - 1 \equiv v + w \pmod{A^t(RG)}. \tag{2}$$

We know that $g_i^{q_i} \in \gamma_t(G)$ holds for all $i = 1, 2, \ldots, s$. Therefore $g_i^{q_i} \in D_t(RG)$ and, by Lemma 3.1, there exist natural numbers $m_i$ for $i = 1, 2, \ldots, s$ such that

$$p_i^{m_i}(g_i - 1) \in A^t(RG). \tag{3}$$

We notice that if $i \leq k$ then $p_i \in \Lambda$. So, for all $i = 1, 2, \ldots, k$, we can decompose the element $r$ from the ideal $\bigcap\limits_{p \in \Lambda} J_p(R)$ as $r = p_i^{m_i} r_i$ $(r_i \in R)$. Therefore $r(g-1) \equiv \sum\limits_{i=1}^{k} r_i p_i^{m_i}(g_i - 1)a_i + rw \pmod{A^t(RG)}$. Then from (2) and (3) we obtain that

$$r(g - 1) \equiv rw \pmod{A^t(RG)}. \tag{4}$$

Now we prove that every component $(g_i - 1)a_i x$ of the sum $wx = \sum\limits_{i=k+1}^{s} (g_i - 1)a_i x$ lies in the ideal $A^t(RG)$. Let $p_j$ be a fixed prime, where $k + 1 \leq j \leq s$. Then $p_j \in \Omega_g \setminus \Lambda$ and from the conditions $x \in \bigcap\limits_{p \in \Omega_g \setminus \Lambda} \bigcap\limits_{n=1}^{\infty} I\left(G^{p^n} \gamma_n(G)\right)$ of our lemma it follows that $x \in \bigcap\limits_{n=1}^{\infty} I\left(G^{q^n} \gamma_n(G)\right)$, where $q = p_j$. For every natural number $n$ we can decompose the element $x$ as $x = \sum\limits_{m=1}^{\ell} \beta_m z_m (f_m - 1)$, where $\beta_m \in R$, $f_m \in G^{q^n} \gamma_n(G)$, and $z_m$ are the elements from a left transversal of the cosets of $G^{q^n} \gamma_n(G)$ in $G$. If $n \geq m_j + t$ then, by Lemma 3.2, we have that

$$f_m - 1 \equiv p_j^{m_j} F_t(f_m) \pmod{A^t(RG)}$$

where $F_t(f_m) \in A(RG)$, and so

$$x \equiv p_j^{m_j} \sum_{m=1}^{\ell} \beta_m z_m F_t(f_m) \pmod{A^t(RG)} \tag{5}$$

holds. Then by (3) we obtain that

$$(g_j-1)a_jx \equiv p_j^{m_j}(g_j-1)a_j\sum_{m=1}^{\ell}\beta_m z_m F_t(f_m) \equiv 0 \pmod{A^t(RG)}$$

i.e. $(g_j-1)a_jx \in A^t(RG)$ for all $j=k+1,\,k+2,\ldots,s$. Thus we have

$$wx = 0 \pmod{A^t(RG)} \tag{6}$$

and

$$rwx = 0 \pmod{A^t(RG)}. \tag{7}$$

If $\Lambda$ is a proper subset of $\Omega_g$ (case 1) then by (4) and (7) we get $r(g-1)x \in A^t(RG)$.

If $\Lambda=\Omega_g$ (case 2) then $w=0$ in (2) and by (4) we obtain that $r(g-1)\in A^t(RG)$.

If $\Lambda=\emptyset$ (case 3) then $v=0$ in (2) and by (6) $(g-1)x\in A^t(RG)$ follows. Because $t$ is arbitrary, from the above facts it follows that the elements $r(g-1)x$, $r(g-1)$ and $(g-1)x$ lie in $A^\omega(RG)$, which proves the lemma. □

Let $G$ be a nilpotent $p$-group of finite exponent. It is clear that its lower central series is an $N$-series.

We now prove the following

**Lemma 3.4.** *For a nilpotent $p$-group $G$ of finite exponent*

$$A^\omega(RG)\subseteq J_p(R)\cdot A(RG).$$

Proof. Let $x=\sum\limits_{i=1}^{n}\alpha_i g_i$ be an element of $RG$ and let $R_p=R/J_p(R)$. Then $J_p(R_p)=0$ and, by Theorem 2.1, $A^\omega(R_pG)=0$. Let $\bar\phi(x)=\sum\limits_{i=1}^{n}\phi(\alpha_i)g_i$ where $\phi$ is the natural homomorphism of $R$ onto $R_p$. Then $\bar\phi$ is a homomorphism of $RG$ onto $R_pG$. If $x\in A^\omega(RG)$ then $\bar\phi(x)$ lies in $A^\omega(R_pG)$. Concequently, $\bar\phi(x)=0$ and $\alpha_i\in J_p(R)$ for $i=1,2,\ldots,n$.

□

Let $\Omega$ be a nonempty subset of the set of primes and let $\mathfrak{N}_p$ be the class of nilpotent $p$-groups of finite exponent. Define $\mathfrak{N}_\Omega$ by $\mathfrak{N}_\Omega=\bigcup\limits_{p\in\Omega}\mathfrak{N}_p$. We have the following

**Theorem 3.1.** *Let $\Omega$ be a nonempty subset of the set of primes such that $\bigcap_{p\in\Omega} J_p(R) = 0$ and a group $G$ is discriminated by the class of groups $\mathfrak{N}_\Omega$. If for every proper subset $\Lambda$ of the set $\Omega$ at least one of the conditions*

1) $\bigcap_{p\in\Lambda} J_p(R) = 0$;
2) *$G$ is discriminated by the class of groups $\mathfrak{N}_{\Omega\setminus\Lambda}$;*

*holds, then $A^\omega(RG) = 0$.*

Proof. Let $x = \sum_{i=1}^{n} \alpha_i g_i \in A^\omega(RG)$ and let $\Lambda$ be the set of those primes of $\Omega$ for which there exists a homomorphism $\phi_p$ of $G$ into the nilpotent $p$-group $H_p$ of finite exponent with the property $\phi_p(g_i) \neq \phi_p(g_j)$ for all $i \neq j$. Since the group $G$ is discriminated by the class of groups $\mathfrak{N}_\Omega$, $\Lambda$ is nonempty.

Let $p$ be an arbitrary element of $\Lambda$. If $\phi_p^* : RG \to RH_p$ is the ring homomorphism, induced by the homomorphism $\phi_p$, then $\phi_p^*(x) = \sum_{i=1}^{n} \alpha_i \phi_p(g_i)$ lies in $A^\omega(RH_p)$. Because $p$ is an arbitrary element of the set $\Lambda$, by Lemma 3.4 we obtain that

$$\alpha_i \in \bigcap_{p\in\Lambda} J_p(R)$$

for all $i = 1, 2, , \ldots, n$. If $\bigcap_{p\in\Lambda} J_p(R) \neq 0$ then from the conditions of this theorem it follows that $G$ is discriminated by the class of groups $\mathfrak{N}_{\Omega\setminus\Lambda}$. Then there exists an element $q \in \Omega \setminus \Lambda$ and a homomorphism $\phi_q$ of the group $G$ into the nilpotent $q$-group of finite exponent $H_q$ for which $\phi_q(g_i) \neq \phi_q(g_j)$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$. Then by construction of the set $\Lambda$ it follows that $q \in \Lambda$. This a contradiction. Consequently, $\bigcap_{p\in\Lambda} J_p(R) = 0$ and $\alpha_i = 0$ for $i = 1, 2, \ldots, n$. Therefore $x = 0$ and $A^\omega(RG) = 0$. □

**Lemma 3.5.** *Let $\mathfrak{G}$ be a class of groups and $\{G_\alpha\}$ $(\alpha \in J)$ a family of the normal subgroups of $G$ such that the conditions*

1) *$G/G_\alpha \in \mathfrak{G}$ for each $\alpha \in J$;*
2) *$G_\alpha$ is torsion-free for all $\alpha \in J$;*

*hold. If $G$ is not discriminated by a class of groups $\mathfrak{G}$ then there exists a finite subset of distinct elements $g_1, g_2, \ldots, g_s$ from $G$ such that the nonzero element $y = (g_1 - 1)(g_2 - 1) \cdots (g_s - 1)$ lies in the ideal $\bigcap_{\alpha\in J} I(G_\alpha)$.*

Proof. If the group $G$ is not discriminated by the class of groups $\mathfrak{G}$ then there exists a finite subset of distinct elements $h_1, h_2, \ldots, h_m$ from $G$

such that for every $\alpha \in J$, $h_iG_\alpha = h_jG_\alpha$ for some $i \neq j$. Then $h_ih_j^{-1} \in G_\alpha$ and the element $h_ih_j^{-1}$ has infinite order.

Let $M = \{g_1, g_2, \dots, g_s\}$ be the set of all those elements of the form $h_ih_j^{-1}$ which have infinite order ($1 \leq i, j \leq m$). It is well known that if $g$ is an element of infinite order then from the equation $(g-1)x = 0$ it follows that $x = 0$ (see for example [7], III. Proposition 4.18). Since $g_i$ ($i$=$1, 2, \dots, s$) have infinite order, the product $y$=$(g_1-1)(g_2-1)\cdots(g_s-1)$ is nonzero. From the construction of the set $M$ it follows that for every $\alpha \in J$ there exists at least one element $g_i$ of $M$ such that $g_i \in G_\alpha$. Concequently, $y \in \bigcap_{\alpha \in J} I(G_\alpha)$. □

**Theorem 3.2.** *Let a group $G$ contain a generalized torsion element. Then $A(RG)$ is residually nilpotent if and only if there exists a nonempty subset $\Omega$ of the set of primes such that $\bigcap_{p\in\Omega} J_p(R) = 0$, the group $G$ is discriminated by the class of groups $\mathfrak{N}_\Omega$, and for every proper subset $\Lambda$ of the set $\Omega$ at least one of the conditions*

1) $\bigcap_{p\in\Lambda} J_p(R) = 0$;
2) *$G$ is discriminated by the calss of groups $\mathfrak{N}_{\Omega\setminus\Lambda}$;*

*holds.*

Proof. Let $A^\omega(RG) = 0$ and let the group $G$ contain a torsion element. Then in $G$ there exists a $p$-element $g$. Let $p^n$ be the order of $g$. We show that in this case the set $\Omega = \{p\}$ satisfies the conditions of this theorem.

Then element $g^{p^n}$ belongs to $\gamma_t(G)$ and $\gamma_t(G) \subseteq D_t(RG)$ for every $t$. Therefore, by Lemma 3.1,

$$p^m(g-1) \in A^t(RG) \tag{8}$$

for some $m$ depending only on $t$. If $r \in J_p(R)$ then for every $m$ the element $r$ can be decomposed as $r = p^m r_m$, $(r_m \in R)$. Therefore from (8) we obtain that $r(g-1) \in A^t(RG)$ for every $t$. Hence $r(g-1) \in A^\omega(RG)$ and $r(g-1) = 0$. This equation is possible only if $r = 0$. Consequently, $J_p(R) = 0$.

Now we how that the group $G$ is discriminated by the class of groups $\mathfrak{N}_p$. Let $1 \neq h \in \bigcap_{n=1}^{\infty} G^{p^n}\gamma_n(G)$. Hence for every $t$ and $m$ there exists $i$ such that $h \in G^{p^i}\gamma_i(G)$ and $i \geq t+m$. Then, by Lemma 3.2,

$$h - 1 \equiv p^m F_t(h) \pmod{A^t(RG)} \tag{9}$$

folows, where $F_t(h) \in A(RG)$. Therefore from (8) we have that $(g-1)\cdot(h-1) \in A^t(RG)$ for all $t$. Then $(g-1)(h-1) \in A^\omega(RG)$ and $(g-1)(h-1) = 0$. This is possible only if $g = h$, $g^2 = 1$, $p = 2$ (because $g$ is a $p$-element of $G$) and the characteristic of the ring $R$ equals 2. Then (9) implies that $h - 1 \in A^t(RG)$ for all $t$. Therefore $h - 1 \in A^\omega(RG)$ and $h = 1$ which is a contradiction. Consequently, $\bigcap\limits_{n=1}^{\infty} G^{p^n}\gamma_n(G) = 1$. From this equation it follows that $G$ is a residually-$\mathfrak{N}_p$ group. Since the class of the groups $\mathfrak{N}_p$ is closed for taking subgroups and forming finite direct products, by Lemma 2.4 we have that $G$ is discriminated by the class of groups $\mathfrak{N}_p$. Concequently, we can choose the set $\Omega$ such that $\Omega = \{p\}$.

Now let $G$ be a torsion-free group with the generalized torsion element $g$ of infinite order. Because $A^\omega(RG) = 0$, from Lemma 2.1 $g \notin \bigcap\limits_{n=1}^{\infty} \gamma_n(G)$ follows. Therefore $\Omega$ is nonempty.

Now we show that the set $\Omega$ can be chosen as $\Omega = \Omega_g$. By Lemma 3.3 (case 2) for every $r \in \bigcap\limits_{p\in\Omega} J_p(R)$ the element $r(g-1)$ lies in $A^\omega(GR)$. Then $r(g-1) = 0$. Therefore $r = 0$ and $\bigcap\limits_{p\in\Omega} J_p(R) = 0$.

Let $\Lambda$ be a subset of $\Omega$. If $G$ is not discriminated by $\mathfrak{N}_{\Omega\setminus\Lambda}$ then, according to Lemma 3.5 (here we suppose that the family $\{G_\alpha\}$ of the normal subgroups of $G$ coincides with $\{G^{p^n}\gamma_n(G),\ p \in \Omega\setminus\Lambda, n = 2, 3, \dots\}$, the class $\mathfrak{G}$ is $\mathfrak{N}_{\Omega\setminus\Lambda}$) there exists a nonzero element $x$ in the ideal $\bigcap\limits_{p\in\Omega_g\setminus\Lambda} \bigcap\limits_{n=1}^{\infty} I(G^{p^n}\gamma_n(G))$ of the form $x = (g_1 - 1)(g_2 - 1)\cdots(g_s - 1)$.

If $\Lambda$ is empty then, by case 3 of Lemma 3.3, we have that $(g-1)x \in A^\omega(RG)$. Therefore $(g-1)x = 0$. In the group ring of torsion-free groups such an equation is impossible. Hence $G$ is discriminated by the class of groups $\mathfrak{N}_\Omega$.

Now let $\Lambda$ be a proper subset of the set $\Omega$ and $r$ be an arbitrary element from $\bigcap\limits_{p\in\Lambda} J_p(R)$. If $G$ is not discriminated by $\mathfrak{N}_{\Omega\setminus\Lambda}$ then, by Lemma 3.3 (case 1), $r(g-1)x \in A^\omega(RG)$. Therefore $r(g-1)x = 0$. This equation is possible only if $r = 0$. Hence if $G$ is not discriminated by the class of groups $\mathfrak{N}_{\Omega\setminus\Lambda}$ then $\bigcap\limits_{p\in\Lambda} J_p(R) = 0$.

Sufficiency is proved in Theorem 3.1. $\square$

If the additive group of a ring $R$ is torsion-free and if a group $G$ has no generalized torsion elements then the question about the residual nilpotency of the augmentation ideal is solved (see in particular [2] Theorem 15.5).

The torsion subgroup $T(R^+)$ of the additive group $R^+$ of a ring $R$ is the direct sum of its primary components $S_p(R^+)$ which are the ideals of the ring $R$.

Let $\Pi$ be the set of those primes for which the $p$-primary components $S_p(R^+)$ of $T(R^+)$ are nonzero.

**Lemma 3.6.** *Let $A^\omega(RG) = 0$ and $T(R^+) \neq 0$. Then $G$ is a residually-$\mathfrak{N}_p$ group for all $p \in \Pi$.*

Proof. Let $p \in \Pi$ and $0 \neq r \in S_p(R^+)$. Then $p^s r = 0$ for some $s$. If $g \in \bigcap_{n=1}^{\infty} G^{p^n}\gamma_n(G)$ then, by Lemma 3.2, we obtain that for all $t$ $g - 1 \equiv p^s F_t(g) \pmod{A^t(RG)}$ ($F_t(g) \in A(RG)$, $n \geq t+s$, $s \geq 1$). Hence $r(g-1) \equiv p^s r F_t(g) \equiv 0 \pmod{A^t(RG)}$ for all $t$ and $r(g-1) \in A^\omega(RG)$). Consequently, $g = 1$ and $\bigcap_{n=1}^{\infty} G^{p^n}\gamma_n(G) = 1$ i.e. $G$ is a residually-$\mathfrak{N}_p$ group.

□

The element $g$ of the additive Abelian group $G$ is called an element of *infinite $p$-height* if the equation $p^n x = g$ has a solution in $G$ for all natural numbers $n$.

**Theorem 3.3.** *Let the torsion group $T(R^+)$ of the additive group $R^+$ of a ring $R$ be nonzero, and suppose that for some $p \in \Pi$ the group $T(R^+)$ has no elements of infinite $p$-height. Further let $G$ be a group with no generalized torsion elements. Then $A^\omega(RG) = 0$ if and only if $G$ is a residually-$\mathfrak{N}_p$ group for all $p \in \Pi$.*

Proof. Let $G$ be a residually-$\mathfrak{N}_p$ group for all $p \in \Pi$ and let $R_p = R/J_p(R)$. Then $J_p(R_p) = 0$ and, by Theorem 2.1, we obtain that $A^\omega(R_p H) = 0$ for all $H \in \mathfrak{N}_p$ and every prime $p \in \Pi$. Therefore, by Lemmas 2.3 and 2.4, we have that $A^\omega(R_p G) = 0$. If the element $x = \sum_{i=1}^{n} \alpha_i g_i$ lies in the ideal $A^\omega(RG)$ then from the last equation we conclude that

$$\alpha_i \in J_p(R) \tag{10}$$

for all $p \in \Pi$ and every $i = 1, 2, \dots, n$.

Let $\sqrt{\gamma_n(G)} = \{g \in G \mid g^m \in \gamma_n(G)$ for some integer $m \geq 1\}$ be the isolator of the subgroup $\gamma_n(G)$ in $G$. Obviously, the $G/\sqrt{\gamma_n(G)}$ are torsion-free nilpotent groups and $\bigcap_{n=1}^{\infty} \sqrt{\gamma_n(G)} = 1$ because the group $G$ has no generalized torsion elements. Consequently, $G \in \mathrm{R}\mathfrak{N}_0$ i.e. $G$ is a residually torsion-free nilpotent group. Since the additive group of the ring $R/T(R^+)$ is without torsion, by Theorem 2.2. $A^\omega(R/T(R^+)G) = 0$

follows. From this we infer that $\alpha_i \in T(R^+)$ and by (10) we obtain that $\alpha_i \in \bigcap_{p\in\Pi} (J_p(R) \cap T(R^+))$ for $i = 1, 2, \dots, n$. It is known that $T(R^+)$ is the serving subgroup of $R^+$. Therefore

$$J_p(R) \cap T(R^+) = \left(\bigcap_{n=1}^{\infty} p^n R\right) \cap T(R^+) = \bigcap_{n=1}^{\infty} \big((p^n R) \cap T(R^+)\big) = \\ = \bigcap_{n=1}^{\infty} p^n T(R^+) = J_p(T(R^+)).$$

Hence $\alpha_i \in \bigcap_{p\in\Pi} J_p(T(R^+))$ for $i = 1, 2, \dots, n$. Because for some $p = p_0 \in \Pi$, the torsion group $T(R^+)$ has no element of infinite $p_0$-height, $J_{p_0}(T(R^+)) = 0$. Therefore $\bigcap_{p\in\Pi} J_p(T(R^+)) = 0$ and $\alpha_i = 0$ for all $i = 1, 2, \dots, n$. Consequently, $x = 0$ and $A^\omega(RG) = 0$.

The necessity is proved in Lemma 3.5. □

*Remark 1.* It is obvious that $T(R^+) \subseteq pT(R^+)$ for all $p \notin \Pi$ and consequently the question about the residual nilpotency of the augmentation ideals remains open in the case when the group $G$ has no generalized torsion elements and the torsion group of the additive group of the ring $R$ has an element of infinite $p$-height for all primes $p$.

*Remark 2.* Theorems 3.1, 3.2 and 3.3 were announced in [4].

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