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Note on the problem of de la Vallée Poussin

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Abstract. A survey of results related to this problem is given. The extension to the vector case is examined. The main stimulation comes from the lecture of Á. Elbert given at the ICNO XII Conference in Cracow 1990 (see also [10]).

1. In reply to the question concerning the lower distance estimate of the consecutive zeros t_1, t_2 of the nontrivial oscillatory solutions $x(t)$ of the equation

$$x'' + a(t)x' + b(t)x = 0,$$

with continuous bounded coefficients on $[t_1, t_2]$, namely

$$A := \max_{t\in[t_1,t_2]} |a(t)|, \quad B := \max_{t\in[t_1,t_2]} |b(t)|,$$

Ch. J. de la Vallée Poussin [24] came for $h = t_2 - t_1 > 0$ to the inequality

$$1 < 2Ah + \frac{1}{2}Bh^2.$$

Since that time there have been stated several improvements of this result, e.g.,

$$1 < \frac{1}{2}Ah + \frac{1}{6}Bh^2,$$

by P. Hartman and A. Wintner [11] or, so far the sharpest inequality of this type,

$$1 \le \frac{2Ah}{\pi^2} + \frac{Bh^2}{\pi^2},$$

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by. Z. Opial [17], [18] as well as the generalization to higher-order equations. For

$$x^{(n)} + \sum_{k=1}^{n} a_k(t)x^{(n-k)} = 0,$$

and $x(t_1) = \cdots = x(t_n) = 0$ $(t_1 < t_2 < \cdots < t_n)$ the two following estimates were obtained for $h = t_n - t_1 > 0$ by A. Yu. Levin [14], [15] (cf. also the similar result in [26] for the special case of the above equation)

$$\sum_{k=1}^{n-1} \frac{h^k}{k!} A_k + \frac{(n-1)^{n-1}}{n^n n!} h^n A_n > 1,$$

$$\sum_{k=1}^{n} \frac{h^k}{2^k k[\frac{1}{2}(k-1)]!(\frac{1}{2}k)!} A_k > 1,$$

where

$$A_k := \max_{t\in[t_1,t_2]} |a_k(t)| \qquad k = 1, \ldots, n.$$

Furthermore, for $n = 3$ A. Lasota [12] and for $n = 4$ D. Bobrowski [7] have established the "Opial-type" inequalities

$$1 < \frac{1}{4}A_1 h + \frac{A_2 h^2}{\pi^2} + \frac{A_3 h^3}{2\pi^2},$$

and

$$1 < \frac{1}{4}A_1 h + \frac{A_2 h^2}{\pi^2} + \frac{A_3 h^3}{2\pi^2} + \frac{A_4 h^4}{2\pi^2}.$$

Nevertheless, until the appearance of the recent paper [8] by J.H. E. Cohn, the problem of the optimal estimate has remained open even for $n = 2$. In [8], the sharpest inequality has been found just for $n = 2$ in the form

$$h \geq 2\int_0^\infty \frac{dt}{1 + At + Bt^2}.$$

2. The main purpose of this note consists (beyond making a survey of results) in performing the appropriate vector extension. Before this we would like to point out that the result in [8] can be generalized to the nonlinear equation

$$x'' + f(t, x, x') = 0,$$

satisfying $|f(t,x,y)| \le A|y| + B|x|$ everywhere, as well as to those for $n = 2,3,4$ in [7], [12], [18], where, moreover, the equation can even take the more general form

$$x^{(n)} + f(t,x,\dots,x^{(n-1)}) = ex^{(n-1)} + g(t,x,\dots,x^{(n-1)}), \tag{0}$$

where e is an arbitrary real constant.

Let us note that in [10] an analog of [8] has been derived for the half-linear equation

$$x'' + a(t)|x'|\operatorname{sgn} x + b(t)x = 0,$$

by extending Sturmian comparison theorems to such differential equations. Moreover for $n = 3$ a nonlinear version of [12] has been already derived in [6], [25].

Following step by step the approach made in [8], one can readily check that the first nonlinear generalization is trivial. Nevertheless, the higher-order analog is not yet known (for some further results in this field see also [5], [9], [16], [19], [22], [23]).

Another indicated nonlinear generalization for $n = 2,3,4$ can be derived by multiplication of (0) with $x^{(n-2)}(t)$, where $x(t)$ is a solution satisfying the oscillatory conditions prescribed, and integration is from u to v, where u, v (implied by these conditions) are such that

$$x^{(n-2)}(u) = x^{(n-2)}(v) = 0 \qquad u < v.$$

Assuming that

$$|f(t,x,\dots,x^{(n-1)})| \le M_0|x| + \cdots + M_{n-1}|x^{(n-1)}|,$$

and $g(t,x,\dots,x^{(n-1)})x^{(n-2)} \ge 0$ everywhere, we obtain the relation

$$\begin{aligned}
&\int_u^v [x^{(n-1)}(t)]^2\,dt = \int_u^v f(t,x,\dots,x^{(n-1)}(t))\,x^{(n-2)}(t)\,dt \\
&- \int_u^v g(t,x(t),\dots,x^{(n-1)}(t))\,x^{(n-2)}(t)\,dt \\
&- \frac{e}{2}\{[x^{(n-2)}(v)]^2 - [x^{(n-2)}(u)]^2\} \\
&\le \int_u^v [M_0|x(t)\,x^{(n-2)}(t)| + \cdots + M_{n-1}|x^{(n-1)}(t)\,x^{(n-2)}(t)|]\,dt.
\end{aligned}$$

Using the "Opial-type" approach, i.e., applying the same integral inequalities as in [7], [12], [18] to the last part of the above relation, we arrive at the desired estimates which are exactly the same as those mentioned already above for linear equations.

3. Now, consider the vector Jacobi-type equation

$$X'' + A(t)X = 0, \tag{1}$$

where $X = (x_1, \dots, x_m)^T, A(t) = (a_{ij}(t))_1^m \in C([a,b])$, i.e., in components

$$x_i'' + \sum_{j=1}^{m} a_{ij}(t)x_j = 0 \qquad i = 1, \dots, m.$$

To (1) we introduce the notion of conjugate points (see [2]–[4]):

Definition. If there exists a nontrivial solution of (1) such that

$$X(a) = 0 = X(b), \tag{2}$$

with $b > a$ and if b is the smallest number larger than a with this property, then b is called the first conjugate point to a relative to (1).

The following statements have been proved in [2]–[4] (cf. also [21]).

Proposition 1. *Let $a_{ij}(t) \geq 0$ for $t \in [a,b]$; $i,j = 1, \dots, m$. If b is the first conjugate point of a (relative to (1)) and if there is a number $c \in (a,b)$ such that $A(c)$ is irreducible (i.e., if there do not exist nonempty subsets I and J of $\{1, \dots, m\}$ such that $I \cap J = \emptyset$, $I \cup J = \{1, \dots, m\}$ and $a_{ij}(c) = 0$ if $i \in I$, $j \in J$), then there exists a solution $X(t)$ of (1) such that (2) is satisfied and*

$$x_i(t) > 0 \quad \text{for} \quad t \in (a,b), \quad i = 1, \dots, m.$$

Proposition 2. *Let $a_{ij}(t) \geq 0$ for $t \in [a,b]$; $i,j = 1, \dots, m$. If b is the first comjugate point of a relative to (1) and if $A(t)$ is symmetric, then there exists a solution $X(t)$ of (1) such that (2) is satisfied and*

$$x_i(t) \geq 0 \quad \text{for} \quad t \in [a,b], \quad i = 1, \dots, m.$$

Moreover observed in [21], each solution $X(t)$ of (1)–(2) can be expressed by means of the nonnegative Green function $G(a,b,t,s)$ as

$$X(t) = \int_a^b G(a,b,t,s)\, A(s)\, X(s)\, ds,$$

i.e., in components

$$x_i(t) = \int_a^b G(a,b,t,s) \sum_{j=1}^{m} a_{ij}(s)\, x_j(s)\, ds \qquad i = 1, \dots, m.$$

Hence, if there is some $k \in \{1, \dots, m\}$ such that

$$a_{kj}(t) \geq a_{ij}(t) \geq 0 \quad \text{for} \quad t \in [a,b]; \qquad i,j = 1, \dots, m, \tag{3}$$

then $X(t)$ above satisfies

$$x_k(t) \geq x_i(t) \quad \text{for} \quad t \in [a,b], \qquad i = 1, \dots, m, \tag{4}$$

as well as

$$x_k(t) > 0 \quad \text{for} \quad t \in (a,b), \tag{5}$$

because otherwise there would be some $c \in (a,b)$ with

$$x_k(c) = x_i(c) = 0 \quad \text{for all} \quad i = 1, \dots, m,$$

which contradicts the assumption that $b(> c)$ is the first conjugate point to a relative to (1).

Our main aim here is to show that, under the assumptions of Proposition 1 or Proposition 2 and (3), the solution $X(t)$ of (1)–(2) exists with

$$h = b - a \geq \frac{\pi}{\sqrt{A_k}}, \quad \text{where} \quad A_k := \max_{t \in [a,b]} \sum_{j=1}^{m} a_{kj}(t). \tag{6}$$

For this purpose we will appropriately modify the approach employed for the scalar case in [8]. Thus, denote by $\alpha_k \in (a,b)$ the first (from left) focal point, $x_k'(\alpha_k) = 0$, and consider on $[a,b]$ the Prüfer-like transformation

$$x_i(t) = r_i(t) \sin \theta_i(t), \quad x_i'(t) = r_i(t) \cos \theta_i(t) \qquad (i = 1, \dots, m).$$

Because of (5) we have also

$$x_k'(t) > 0 \quad \text{for} \quad t \in (a, \alpha_k),$$

and consequently we can assume without any loss of generality that

$$r_k(t) > 0 \quad \text{and} \quad 0 \leq \theta_k(t) \leq \frac{\pi}{2} \quad \text{for} \quad t \in [a, \alpha_k],$$

where

$$\theta_k(a) = 0 \quad \text{and} \quad \theta_k(\alpha_k) = \frac{\pi}{2}.$$

So, we come to

$$x_k'' = r_k' \cos \theta_k - r_k \theta_k' \sin \theta_k = -\sum_{j=1}^{m} a_{kj} r_j \sin \theta_j,$$
$$x_k' = r_k' \sin \theta_k + r_k \theta_k' \cos \theta_k = r_k \cos \theta_k.$$

Eliminating r_k', we get

$$r_k \theta_k' = r_k \cos^2 \theta_k + \sin \theta_k \sum_{j=1}^{m} a_{kj} r_j \sin \theta_j.$$

Because of (4) we have also

$$r_k(t)\sin\theta_k(t) \geq r_i(t)\sin\theta_i(t) \quad \text{for} \quad t \in [a,b], \qquad i = 1,\dots,m,$$

so that

$$r_k\theta_k' \leq r_k(\cos^2\theta_k + \sin^2\theta_k \sum_{j=1}^{m} a_{kj}).$$

Dividing the last inequality by $r_k(t)$ (> 0 for $t \in [a,\alpha_k]$), we arrive at

$$\theta_k' \leq \cos^2\theta_k + A_k \sin^2\theta_k,$$

where A_k is defined in (6).

Therefore,

$$\begin{aligned}\alpha_k - a &\geq \int_0^{\frac{\pi}{2}} \frac{d\theta_k}{\cos^2\theta_k + A_k\sin^2\theta_k} = \int_0^{\infty} \frac{dt}{1+A_kt^2} \\ &= \left[\frac{1}{\sqrt{A_k}}\operatorname{arctg}\sqrt{A_k}t\right]_0^{\infty} = \frac{\pi}{2\sqrt{A_k}}.\end{aligned}$$

Since we can obtain quite analogously that $b - \beta_k \geq \frac{1}{2}\pi\sqrt{A_k}$, where $\beta_k \in (a,b)$ is the last (first from right) focal point, $x_k'(\beta_k) = 0$, we can finally conclude that $b - a \geq (\alpha_k - a) + (b - \beta_k) \geq \frac{\pi}{\sqrt{A_k}}$, i.e., (6).

Theorem. *Under the assumptions of Proposition 1 or Proposition 2 and* (3)*, there exists a solution of* (1) *satisfying* (2)*, such that* (6) *holds.*

4. *Remark 1.* For the scalar equation $x'' + b(t)x = 0$, where $b(t) > 0$, the same (i.e., $h = t_2 - t_1 \geq \pi/\sqrt{B}$, where $B := \max_{t\in[t_1,t_2]} b(t)$) can be proved for the lower distance estimates of zero points of the derivatives (focal points), using the transformation $x = r(t)\cos\theta(t)$, $x' = r(t)\sin\theta(t)$.

Remark 2. We believe that the best result for the vector case might be obtained by means of the abstract Prüfer transformation. The idea of the generalized concept of Prüfer's transformation is due to J.H. Barrett; for details see e.g., the monograph [20], where many appropriate references can also be found. Unfortunately, we have not yet been able to find it.

Remark 3. As it has been pointed out in [13], the solution of the homogeneous linear de la Vallée Poussin problem can be directly applied to the one of the (non)homogeneous (non)linear interpolation problem, i.e., when the desired solution satisfies the prescribed values in n points.

Indeed, it is enough to replace the desired inequality for h by the converse one, i.e., for $n = 3$, by (cf. above)

$$1 > \frac{A_1 h}{4} + \frac{A_2 h^2}{\pi^2} + \frac{A_3 h^3}{2\pi^2}.$$

This is not, however, the best known sufficient condition. Consider, for example, the equidistant case with

$$1 > \frac{3}{16} A_1 h + \frac{33}{1280} A_2 h^2 + \frac{2}{1280} A_3 h^3,$$

dealt with in [1], where the best possibility is also discussed in detail.

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