

Homogeneous summands of exponentials

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Introduction

As usual $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ the integers, and $\mathbf{T}$ the unit circle. Let n be a positive integer, $\mathbb{Z}_n$ the group of residues modulo n , and ω a primitive n 'th root of unity.

According to [3, Th. 3.1], any function $f : \mathbb{C} \rightarrow \mathbb{C}$ is uniquely the sum of functions, $f = \sum_{j \in \mathbb{Z}_n} f_j$, such that

$$f_j(\omega x) = \omega^j f_j(x) \quad x \in \mathbb{C} \text{ and } j \in \mathbb{Z}_n.$$

Since $\omega^n = 1$, no confusion results from writing ω^j even if $j \in \mathbb{Z}_n$. The summand f_j is said to be of type j . For instance, with $n = 2$, this describes the decomposition of a function into its even and odd parts. If f is entire with series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f_j(x) = \sum_{k \equiv j \pmod{n}} a_k x^k$, where $x \in \mathbb{C}$ and $j \in \mathbb{Z}_n$.

An exponential is a function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$(1) \quad f(x+y) = f(x)f(y) \quad x, y \in \mathbb{C}.$$

The type j summands of an exponential satisfy the system of equations

$$(2) \quad f_j(x + \omega y) = \sum_{k \in \mathbb{Z}_n} \omega^{j-k} f_k(x) f_{j-k}(y) \quad x, y \in \mathbb{C}, \quad j \in \mathbb{Z}_n.$$

Furthermore it was demonstrated in [3, Th. 3] that any n -tuple of functions $(f_j)_{j \in \mathbb{Z}_n}$, satisfying (2), arises from a unique exponential f according to

$$(3) \quad f_j(x) = \frac{1}{n} \sum_{k \in \mathbb{Z}_n} \omega^{jk} f(\omega^{-k} x) \quad x \in \mathbb{C}, \quad j \in \mathbb{Z}_n.$$

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In this note we consider a situation which is the analogue of the one above when n is replaced by ∞ . Here $\mathbb{Z}_n$ gets replaced by $\mathbb{Z}$, and the n 'th roots of unity by $\mathbf{T}$. Finite sums become absolutely convergent series.

Homogeneous summands

It seems natural to say that a function $f : \mathbb{C} \rightarrow \mathbb{C}$ is absolutely decomposable into homogeneous summands provided, for each integer n , there is a function $f_n : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$(4) \quad f_n(\sigma x) = f_n(x)\sigma^n \quad \sigma \in \mathbf{T}, x \in \mathbb{C}$$

and

$$(5) \quad f(x) = \sum_{n \in \mathbb{Z}} f_n(x) \quad \text{absolutely} \quad x \in \mathbb{C}$$

The set of such f will be denoted by $\mathcal{D}$.

Proposition 1. *A function f belongs to $\mathcal{D}$ if and only if, for each positive real r , the function $f \cdot r : \mathbf{T} \rightarrow \mathbb{C}$, given by $f \cdot r(\xi) = f(r\xi)$, $\xi \in \mathbf{T}$, has absolutely summable Fourier coefficients.*

PROOF. If f in $\mathcal{D}$ decomposes as $f = \sum_{n \in \mathbb{Z}} f_n$, then

$$(f \cdot r)(\xi) = \sum_{n \in \mathbb{Z}} f_n(r\xi) = \sum_{n \in \mathbb{Z}} f_n(r)\xi^n \quad r > 0, \xi \in \mathbf{T}.$$

Since this series converges absolutely, the values $f_n(r)$ give the absolutely summable Fourier coefficients of $f \cdot r$.

Conversely suppose for each $r > 0$ that $f \cdot r$ has absolutely summable Fourier coefficients $(f_n(r))_{n \in \mathbb{Z}}$. If $x \in \mathbb{C}$ and $x \neq 0$, then $x = r\xi$ for a unique $r > 0$ and unique ξ in $\mathbf{T}$; and

$$f(x) = f(r\xi) = (f \cdot r)(\xi) = \sum_{n \in \mathbb{Z}} f_n(r)\xi^n$$

with the convergence absolute. For every integer n define $f_n : \mathbb{C} \rightarrow \mathbb{C}$ by $f_n(x) = f_n(r)\xi^n$ when $x = r\xi$, $r > 0$, $\xi \in \mathbf{T}$; and by $f_n(0) = \delta_{0,n}f(0)$, where δ is the Kronecker delta. Each f_n satisfies (4), and $f = \sum_{n \in \mathbb{Z}} f_n$ as in (5). Thus $f \in \mathcal{D}$. $\square$

The above proposition gives the recipe for constructing all possible decomposable functions. Because each $f \cdot r$ is continuous, it also implies that each f in $\mathcal{D}$, restricted to any circle centred at the origin, must be continuous.

Proposition 2. *If $f \in \mathcal{D}$, then the homogeneous summands of f are unique.*

PROOF. It suffices to see that the zero function $\mathbf{0}$ has a unique decomposition. If $\mathbf{0} = \sum_{n \in \mathbb{Z}} f_n$ absolutely with f_n homogeneous, then $\sum_{n \in \mathbb{Z}} f_n(r)\xi^n = 0$ for every $r > 0$ and ξ in $\mathbf{T}$. This forces $f_n(r) = 0$ due to [2, Sec. 6.1], and thus $f_n(x) = 0$ when $x \neq 0$. When $x = 0$ the requirement $f_n(0) = f_n(\sigma 0) = f_n(0)\sigma^n$, for each n in $\mathbb{Z}$ and σ in $\mathbf{T}$, yields $f_n(0) = \delta_{0,n}\mathbf{0}(0) = 0$. Thus the homogeneous summands for the zero function must all be zero. $\square$

The space $\ell^1 = \ell^1(\mathbb{Z})$ of absolutely summable complex bilateral sequences is a commutative algebra under convolution:

$$(x_n)_{n \in \mathbb{Z}} * (y_n)_{n \in \mathbb{Z}} = \left(\sum_{k \in \mathbb{Z}} x_k y_{n-k} \right)_{n \in \mathbb{Z}} \quad (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \in \ell^1.$$

The set $\mathcal{F}$ of all maps $\mathbb{C} \rightarrow \ell^1$ inherits this algebra structure under pointwise operations. In particular for $F, G : \mathbb{C} \rightarrow \ell^1$ in $\mathcal{F}$ the product $F * G$ is given by

$$(F * G)(x) = F(x) * G(x) \quad x \in \mathbb{C}.$$

Each F in $\mathcal{F}$ has component maps $f_n : \mathbb{C} \rightarrow \mathbb{C}$, i.e. $F(x) = (f_n(x))_{n \in \mathbb{Z}}$, $x \in \mathbb{C}$. The set $\mathcal{A}$ of those F such that their components f_n satisfy (4) constitutes a subalgebra of $\mathcal{F}$. To see this, let $F = (f_n)_{n \in \mathbb{Z}}$, $G = (g_n)_{n \in \mathbb{Z}} \in \mathcal{A}$, and let $H = F * G$. If $x \in \mathbb{C}$ and $H(x) = (h_n(x))_{n \in \mathbb{Z}}$, then $h_n(x) = \sum_{k \in \mathbb{Z}} f_k(x)g_{n-k}(x)$. For σ in $\mathbf{T}$

$$\begin{aligned} h_n(\sigma x) &= \sum_{k \in \mathbb{Z}} f_k(\sigma x)g_{n-k}(\sigma x) = \sum_{k \in \mathbb{Z}} f_k(x)\sigma^k g_{n-k}(x)\sigma^{n-k} \\ &= \left(\sum_{k \in \mathbb{Z}} f_k(x)g_{n-k}(x) \right) \sigma^n = h_n(x)\sigma^n. \end{aligned}$$

Thus $H \in \mathcal{A}$.

Proposition 3. *Under pointwise addition and multiplication the set $\mathcal{D}$ of all decomposable functions $f : \mathbb{C} \rightarrow \mathbb{C}$ is an algebra isomorphic to $\mathcal{A}$.*

PROOF. The isomorphism $\mathcal{A} \rightarrow \mathcal{D}$ is given by $F = (f_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} f_n$. Proposition 2 ensures that this is a bijection. Pointwise convolution in $\mathcal{A}$ corresponds to multiplication in $\mathcal{D}$ because absolutely summable series obey the distributive law, and the order of summation is immaterial. $\square$

Proposition 4. *If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a continuous exponential, then $f \in \mathcal{D}$.*

PROOF. If f is the zero function, then $f \in \mathcal{D}$ for trivial reasons. Otherwise, as is well known (see e.g. [1, Ch. 5, Th. 3]), there exist a, b in $\mathbb{C}$ such that

$$f(x) = e^{ax+b\bar{x}} = e^{ax}e^{b\bar{x}} \quad x \in \mathbb{C}.$$

Let $r > 0$, $\xi \in \mathbf{T}$ and $x = r\xi$. Then

$$e^{ax} = e^{ar\xi} = \sum_{n \geq 0} \frac{a^n r^n}{n!} \xi^n, \quad e^{b\bar{x}} = e^{br\xi^{-1}} = \sum_{n \geq 0} \frac{b^n r^n}{n!} \xi^{-n}.$$

Since the coefficients $a^n r^n/n!$ and $b^n r^n/n!$ are absolutely summable, Proposition 1 yields that the maps $x \mapsto e^{ax}$, $x \mapsto e^{b\bar{x}}$ are in $\mathcal{D}$. Their product f is in $\mathcal{D}$, from Proposition 3. $\square$

Decompose a continuous exponential f as $\sum_{n \in \mathbb{Z}} f_n$ where f_n satisfy (4). Since f satisfies (3), it follows from Proposition 3 that

$$(6) \quad f_n(x+y) = \sum_{k \in \mathbb{Z}} f_k(x)f_{n-k}(y) \quad x, y \in \mathbb{C}, \quad n \in \mathbb{Z}.$$

Equations (4) and (6), for the homogeneous summands of a continuous exponential, seem to be the natural extension of (2) to the limiting case of ∞ . The point of this note is that, as in [3, Th. 3], an exponential f , now continuous, can be recovered from (4) and (6).

Theorem 5. *Suppose $F = (f_n)_{n \in \mathbb{Z}} \in \mathcal{A}$ and $F(x+y) = F(x) * F(y)$ for x, y in $\mathbb{C}$; that is, f_n satisfy (4) and (6) as well as absolute summability. Then the map $f = \sum_{n \in \mathbb{Z}} f_n$ is a continuous exponential with the f_n 's as its homogeneous components.*

PROOF. From Proposition 3 it is clear that f is an exponential and that the f_n 's give its homogeneous components. There remains the question of the continuity of f .

As observed after Proposition 1, the restriction of f to any circle centred at the origin is continuous. By considering $f(x+y) = f(x)f(y)$ where $x, y \in \mathbb{C}$, fixing y and letting x vary, it is seen that the restriction of f to a circle with any centre is continuous. This consideration also shows that, if f is continuous at one point in $\mathbb{C}$, then f is continuous at all points. Thus it suffices to prove the continuity of f at the point $1+i$, $i = \sqrt{-1}$.

Consider the smooth map φ , of two real variables, given by

$$\varphi : s + it \mapsto \left(s + 1 - \sqrt{2t - t^2} \right) + i \left(t + 1 - \sqrt{2s - s^2} \right) \quad 0 < s, t < 2.$$

A simple verification shows that $\varphi(1+i) = 1+i$, and the Jacobian matrix of φ at $1+i$ is the 2×2 identity matrix. By the inverse function theorem, the restriction of φ to some open neighbourhood V of $1+i$ is an open mapping and injective. Since $\varphi(V)$ is now another open neighbourhood of $1+i$, homeomorphic to V by φ , f becomes continuous at $1+i$ provided $f \circ \varphi$ restricted to V is continuous at $1+i$.

Suppose $s_n + it_n$ is a sequence in V and $s_n + it_n \rightarrow 1$ as $n \rightarrow \infty$, i.e. $s_n \rightarrow 1$, $t_n \rightarrow 1$. Let

$$u_n = s_n + i \left(1 - \sqrt{2s_n - s_n^2}\right), \quad v_n = 1 - \sqrt{2t_n - t_n^2} + it_n.$$

Clearly $u_n \rightarrow 1$, $v_n \rightarrow i$ and $u_n + v_n = \varphi(s_n + it_n)$. Now observe that $|1+i-u_n| = |1+i-v_n| = 1$, which means that u_n, v_n lie on the circle K of centre $1+i$ and radius 1. The exponential f when restricted to such K , is continuous. Thus $f(u_n) \rightarrow f(1)$, $f(v_n) \rightarrow f(i)$ as $n \rightarrow \infty$. This leads to the conclusion that

$$\begin{aligned} f \circ \varphi(s_n + it_n) &= f(u_n + v_n) = f(u_n)f(v_n) \rightarrow \\ &= f(1)f(i) = f(1+i) = f \circ \varphi(1+i), \end{aligned}$$

as desired. $\square$

The above proof leads one to speculate, out of mere curiosity, whether any function $\mathbb{C} \rightarrow \mathbb{C}$, having continuous restrictions to all circles in $\mathbb{C}$, can still possess a point of discontinuity.

What is the analogue of (3) when n is replaced by ∞ ? Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is a continuous exponential; and thus $f = \sum_{n \in \mathbb{Z}} f_n$ absolutely such that the f_n 's satisfy (4) and (6). If $x \in \mathbb{C}$ and $\sigma \in \mathbf{T}$, then $f(\sigma x) = \sum_{n \in \mathbb{Z}} f_n(x) \sigma^n$. Since the coefficients of σ^n are absolutely summable, this series converges uniformly over all σ in $\mathbf{T}$. For every integer j , term by term integration of $\sigma^{-j} f(\sigma x)$ over σ in $\mathbf{T}$ gives the summand f_j in terms of the exponential. Namely

$$\frac{1}{2\pi} \int_{\mathbf{T}} \sigma^{-j} f(\sigma x) d\sigma = \sum_{n \in \mathbb{Z}} f_n(x) \frac{1}{2\pi} \int_{\mathbf{T}} \sigma^{n-j} d\sigma = f_j(x),$$

for every x in $\mathbb{C}$. The analogue of the summation (3) is the familiar computation of Fourier coefficients over $\mathbf{T}$, the dual group of $\mathbb{Z}$.

References

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