

A condition for a Finsler space to be a Riemannian space

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0. Introduction

In a Finsler space, there are known canonical Finsler connections, that is, the Cartan connection CT , the Berwald connection $B\Gamma$, the Rund connection $R\Gamma$ and the Hashiguchi connection $H\Gamma$.

The concept of the generalized Berwald P^1 -connections was introduced by AIKOU and HASHIGUCHI [1] and MATSUMOTO [2]. In the previous paper [4], we considered a generalized Berwald P^1 -connection Γ . Using this Finsler connection, we obtained a condition for a Finsler space to be a Riemannian space in each of the above-mentioned Finsler connections CT , $B\Gamma$, $R\Gamma$, $H\Gamma$ and Γ , respectively [4].

The purpose of the present paper is to improve these conditions for a Finsler space to be a Riemannian space and to make up these conditions into a condition valid for any of the Finsler connections mentioned above.

Throughout the present paper, the terminology and notations are those of MATSUMOTO's monograph [3].

1. Preliminaries

Let $F^n = (M, L)$ be a Finsler space with a Finsler metric $L(x, y)$, where x denotes a point of the underlying manifold M and y denotes a supporting element. The fundamental tensor g_{ij} is given by $g_{ij} = \frac{1}{2} \partial^2 L^2 / \partial y^i \partial y^j$. We shall express a Finsler connection $F\Gamma(F_h^i{}_j, N_j^i, C_h^i{}_j)$ in terms of its coefficients. The $(v)hv$ -torsion tensor P_{jk}^i and the hv -curvature tensor $P_h^i{}_{jk}$ of a Finsler connection $F\Gamma$ are given by

$$(1.1) \quad P_{jk}^i = \dot{\partial}_k N_j^i - F_k^i{}_j,$$

and

$$(1.2) \quad P_h^i{}_{jk} = \dot{\partial}_k F_h^i{}_{|j} - C_h^i{}_{k|j} + C_h^i{}_{|r} P^r{}_{jk},$$

respectively, where “ $|$ ” denotes the h -covariant differentiation with respect to $F\Gamma$ and $\dot{\partial}_k = \partial/\partial y^k$.

As well-known, the Berwald connection $B\Gamma$ is given by $B\Gamma = (G_h^i{}_{|j}, G^i{}_{|j}, 0)$, where

$$\begin{aligned} G_h^i{}_{|j} &= \frac{1}{2} \dot{\partial}_h \dot{\partial}_j (\gamma_m^i{}_{|r} y^m y^r), \\ \gamma_m^i{}_{|r} &= \frac{1}{2} g^{ij} (\partial g_{jm} / \partial x^r + \partial g_{jr} / \partial x^m - \partial g_{mr} / \partial x^j), \\ G^i{}_{|j} &= G_h^i{}_{|j} y^h = \frac{1}{2} \dot{\partial}_j (\gamma_m^i{}_{|r} y^m y^r). \end{aligned}$$

Then the known Finsler connections $B\Gamma$, $H\Gamma$, $R\Gamma$, $C\Gamma$ and our Finsler connections Γ and Γ' are denoted by the following table:

Finsler connection	$F\Gamma$	$F_h^i{}_{ j}$	$N^i{}_{ j}$	$C_h^i{}_{ j}$
Berwald	$B\Gamma$	$G_h^i{}_{ j}$	$G^i{}_{ j}$	0
Hashiguchi	$H\Gamma$	$G_h^i{}_{ j}$	$G^i{}_{ j}$	$g_h^i{}_{ j}$
Rund	$R\Gamma$	$G_h^i{}_{ j} - g_h^i{}_{ j 0}$	$G^i{}_{ j}$	0
Cartan	$C\Gamma$	$G_h^i{}_{ j} - g_h^i{}_{ j 0}$	$G^i{}_{ j}$	$G_h^i{}_{ j}$
	Γ	$G_h^i{}_{ j} + L g_h^i{}_{ j}$	$G^i{}_{ j}$	0
	Γ'	$G_h^i{}_{ j} + L g_h^i{}_{ j}$	$G^i{}_{ j}$	$g_h^i{}_{ j}$

where $g_h^i{}_{|j} = \frac{1}{2} g^{ir} \dot{\partial}_r g_{hj}$ and $g_h^i{}_{|j|0} = g_h^i{}_{|j|k} y^k$.

For the above-mentioned Finsler connections, the following equations are satisfied:

$$(1.3) \quad F_h^i{}_{|j} = G_h^i{}_{|j} - P^i{}_{jh},$$

$$(1.4) \quad P_0^i{}_{jk} := P_h^i{}_{jk} y^h = P^i{}_{jk}.$$

Using (1.3), (1.2) is rewritten as follows:

$$(1.5) \quad P_h^i{}_{jk} = G_h^i{}_{|j \cdot k} - P^i{}_{hj \cdot k} - C_h^i{}_{k|j} + C_h^i{}_{|r} P^r{}_{jk},$$

where $\cdot k = \dot{\partial}_k$.

2. The main result

In this section, we shall assume that the Finsler connection $F\Gamma$ is one of the Finsler connections $B\Gamma$, $H\Gamma$, $R\Gamma$, $C\Gamma$, Γ and Γ' .

Theorem. *Let $A_h^i{}_j = Lg_h^i{}_j$. Then a Finsler space F^n is Riemannian, if and only if*

$$(2.1) \quad P_h^i{}_{jk} = -A_h^i{}_{j \cdot k}.$$

To prove the theorem, we shall prove the following

Lemma. *Let F^n be a Finsler space with the hv -curvature tensor $P_h^i{}_{jk}$ satisfying (2.1). Then F^n is a Landsberg space satisfying*

$$(2.2) \quad P_j^i{}_{k} = A_j^i{}_{\cdot k}.$$

PROOF of the Lemma. Contracting (2.1) by y^h and paying attention to (1.4), we have (2.2). Thus, from (1.5), (2.1) and (2.2), we get

$$-A_h^i{}_{j \cdot k} = G_h^i{}_{j \cdot k} - A_h^i{}_{j \cdot k} - C_h^i{}_{k|j} + C_h^i{}_{\cdot r} A_j^r{}_{\cdot k}.$$

Hence we have

$$(2.3) \quad G_h^i{}_{j \cdot k} - C_h^i{}_{k|j} + C_h^i{}_{\cdot r} A_j^r{}_{\cdot k} = 0.$$

Case 1. If $F\Gamma$ is $H\Gamma$ or $C\Gamma$ or Γ' , then we have $C_h^i{}_{\cdot j} = g_h^i{}_{\cdot j}$. In this case, contracting (2.3) by y^j , we have $g_h^i{}_{k|0} = 0$. This means that F^n is a Landsberg space.

Case 2. If $F\Gamma$ is $B\Gamma$ or $R\Gamma$ or Γ , then we have $C_h^i{}_{\cdot j} = 0$. Hence (2.3) reduces to $G_h^i{}_{j \cdot k} = 0$. This means that F^n is a Berwald space and hence a Landsberg space. Thus the lemma is proved.

PROOF of the Theorem. The necessity of (2.1) is evident. Assume (2.1), then by the Lemma, we have that F^n is a Landsberg space satisfying (2.2). First, we shall consider cases of $H\Gamma$ and $B\Gamma$. For $H\Gamma$ and $B\Gamma$, the $(v)hv$ -torsion tensor $P^i{}_{jk}$ vanishes. So from (2.2) we have $A_j^i{}_{\cdot k} = P^i{}_{jk} = 0$. Accordingly, F^n is a Riemannian space. Next, we shall consider cases of Γ and Γ' . For these Finsler connections we have $P^i{}_{jk} = -A_j^i{}_{\cdot k}$. So we have $A_j^i{}_{\cdot k} = P^i{}_{jk} = -A_j^i{}_{\cdot k}$. Hence $A_j^i{}_{\cdot k} = 0$. This means that F^n is Riemannian. Third, we shall consider cases of $C\Gamma$ and $R\Gamma$. For these Finsler connections, we have $P^i{}_{jk} = g_j^i{}_{k|0}$. So, noting that F^n is a Landsberg space, we obtain $A_j^i{}_{\cdot k} = P^i{}_{jk} = g_j^i{}_{k|0} = 0$. This means that F^n is Riemannian. Thus the proof of the Theorem is complete.

References

- [1] T. AIKOU and M. HASHIGUCHI, On generalized Berwald connections, *Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.)* **17** (1984), 9–13.
- [2] M. MATSUMOTO, The induced and intrinsic Finsler connections of a hypersurface and Finsler projective geometry, *J. Math. Kyoto Univ.* **25** (1985), 107–144.
- [3] M. MATSUMOTO, Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press, Otsu, Japan*, 1986.
- [4] S. WATANABE, The generalized Berwald P^1 -connection with $P^i_{jk} = -A_j^i{}_{;k}$ and its applications, *Publ. Math. Debrecen* **42** (1993), 39–44.

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