

Real hypersurfaces of e -($J^4 = 1$)-Kähler manifolds

By AUREL BEJANCU (Iași) and GUSTAVO SANTOS-GARCÍA (Salamanca)

§0. Introduction

($J^4 = 1$)-Kähler manifolds were introduced in [3] as a natural generalization of both Kähler manifolds and para-Kähler manifolds. Several interesting results on the topic can be found in [3–5, 7].

We shall consider e -($J^4 = 1$)-Kähler manifolds. As the metric in that case is necessarily semi-Riemannian, the theory of submanifolds has to deal with both degenerate and non-degenerate submanifolds. Such a study is initiated in the present paper for real hypersurfaces of e -($J^4 = 1$)-Kähler manifolds. First we obtain the geometric structure induced on a real hypersurface of an e -metric ($J^4 = 1$)-manifolds. Then we give characterizations of the integrable distributions on a real hypersurface of an e -($J^4 = 1$)-Kähler manifold by means of their second fundamental forms.

§1. Preliminaries

Let M be a real $2m$ -dimensional differentiable manifold endowed with a semi-Riemannian metric g and a tensor field J of type (1,1) satisfying

$$g \circ (J \times I) + g \circ (I \times J) = 0; \quad J^4 = I,$$

where I is the identity map on TM . The characteristic polynomial of J is supposed to be $P_J(\lambda) = (\lambda^2 - 1)^r (\lambda^2 + 1)^s$, with $r + s = m$. Then (M, g, J) is called an e -metric ($J^4 = 1$)-manifold (cf. [3]). If moreover $\nabla J = 0$, where ∇ is the Levi-Civita connection on M with respect to g , then (M, g, J) is called an e -($J^4 = 1$)-Kähler manifold. The letter e appears in these names

as a consequence of the fact that the fundamental 2-form $F = g \circ (I \times J)$ of M , in the case $m = 2$, is related to the electromagnetic field.

Throughout the paper we shall denote by $\Gamma(E)$ the $F(M)$ -module of differentiable sections of a vector bundle E over M , where $F(M)$ is the algebra of differentiable functions on M . We shall use $\oplus$ and $\perp$ for the Whitney sum and the orthogonal Whitney sum of vector bundles, respectively.

Consider the tensor fields $P = \frac{1}{2}(I + J^2)$ and $Q = \frac{1}{2}(I - J^2)$ and express the tangent bundle of M as follows: $TM = U \perp V$, where $U = \text{Im } P$ and $V = \text{Im } Q$. It is easy to check that U and V are para-holomorphic and holomorphic distributions respectively, i.e., J acts as an almost product operator on U and an almost complex operator on V and satisfies

$$g \circ (J \times J) = -g \circ (I \times I) \quad \text{and} \quad g \circ (J \times J) = g \circ (I \times I)$$

on U and V , respectively. Moreover, U and V are orthogonal distributions and (see [3]) M is locally the product of a $2r$ -dimensional para-Kaehler manifold and a $2s$ -dimensional Kaehler manifold, provided that M is an e -($J^4 = 1$)-Kaehler manifold.

Let N be a real hypersurface on an e -metric ($J^4 = 1$)-manifold (M, g, J) . For any $x \in N$ consider the perp-vector space $T_x N^\perp$ to $T_x N$ in $T_x M$ (cf. O'NEILL [6]) and construct the *perp-vector bundle* $TN^\perp = \bigcup_{x \in N} T_x N^\perp$ over N . As our metric is necessarily semi-Riemannian, the induced metric on N , denoted by the same letter g , is either non-degenerate or degenerate according as the perp-vector bundle is non-degenerate or degenerate, respectively.

First we consider the case in which g is non-degenerate and call N a *non-degenerate real hypersurface* of M . In this situation the perp-vector bundle is just the normal bundle of N . Suppose ξ is a unit vector field normal to N and put

$$(1.1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi$$

and

$$(1.2) \quad \bar{\nabla}_X \xi = -A_\xi X,$$

for any $X, Y \in \Gamma(TN)$, where ∇ and $\bar{\nabla}$ are the Levi-Civita connections on N and M respectively, h is the second fundamental form of N and A_ξ is the shape operator of N .

Next, suppose that the induced metric is degenerate on TN , i.e., there exists $\xi \in \Gamma(TN)$, $\xi \neq 0$, such that $g(\xi, X) = 0$, for any $X \in \Gamma(TN)$.

Then we call N a *degenerate (null) real hypersurface* of M . In this case the perp-vector bundle $TN^\perp$ becomes a distribution on N and thus the above theory of non-degenerate real hypersurfaces is useless. In order to study the geometry of the degenerate immersion of N in M we need a vector bundle transversal to TN in TM . The construction of such a vector bundle was performed by BEJANCU and DUGGAL [2] for degenerate hypersurfaces of semi-Riemannian manifolds as follows. First, consider the so-called *screen distribution* SN on N , which is a distribution complementary of $TN^\perp$ in TN . It is proved in [2] that there exists a unique vector bundle tN with rank $tN = 1$ such that for any $\xi \in \Gamma(TN^\perp)$ there exists a unique $\bar{\xi} \in \Gamma(tN)$ satisfying

$$(1.3) \quad g(\bar{\xi}, \bar{\xi}) = g(\bar{\xi}, X) = 0, \quad g(\xi, \bar{\xi}) = 1, \quad \forall X \in \Gamma(SN).$$

In this case, Gauss and Weingarten equations for the degenerate immersion of N in M become

$$(1.4) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)\bar{\xi}$$

and

$$(1.5) \quad \bar{\nabla}_X \bar{\xi} = -A'_\xi X + \tau(X)\bar{\xi},$$

for any $X, Y \in \Gamma(TN)$, where B is the second fundamental form of the immersion, ∇ is a torsion-free connection induced by $\bar{\nabla}$ on N , τ is a 1-form locally defined on N and A'_ξ is the shape operator of N . We have to note that B does not depend on the screen distribution and, in general, ∇ is not a metric connection. In this theory we need also the equations [2]:

$$(1.6) \quad \nabla_X hY = \bar{\nabla}_X^* hY + C(X, hY)\xi$$

and

$$(1.7) \quad \nabla_X \xi = -A_\xi^* X - \tau(X)\xi,$$

for any $X, Y \in \Gamma(TN)$, where h is the projection of TN on SN , $\bar{\nabla}^*$ is the linear connection induced by ∇ on SN , C is an $F(M)$ -bilinear form on $TN \times SN$ and A_ξ^* is a linear operator on $\Gamma(TN)$. Note that, in general, C is not symmetric on $SN \times SN$ but it satisfies

$$(1.8) \quad C(X, hY) = g(A'_\xi X, hY), \quad \forall X, Y \in \Gamma(TN).$$

Finally, note that B is symmetric and satisfies

$$B(X, hY) = g(A_\xi^* X, hY), \quad \forall X, Y \in \Gamma(TN),$$

and

$$B(\xi, Y) = 0, \quad \forall Y \in \Gamma(TN).$$

§2. Geometric structure induced on a real hypersurface on an e -metric ($J^4 = 1$)-manifold

2.1. The case of a non-degenerate real hypersurface.

First, suppose that N is a non-degenerate real hypersurface of (M, g, J) . In order to get the geometric structure induced on N we have to analyse two cases.

Case 2. I. 1. ($TN^\perp$ is a *vector sub-bundle* of U or of V). In this case $J^2(TN^\perp) = TN^\perp$ and therefore $J(TN^\perp) = J^3(TN^\perp)$. As $J(TN^\perp)$ is orthogonal to $TN^\perp$ it follows that the tangent bundle of N decomposes as follows:

$$(2.1) \quad TN = J(TN^\perp) \perp D_1,$$

where D_1 is the orthogonal distribution complementary of $J(TN^\perp)$ in TN . Moreover, it is easy to check that D_1 is J -invariant, and therefore N is the analogous of a CR-submanifold for Kaehler manifolds (cf. [1]).

Case 2. I. 2. ($TN^\perp \cap U = \{0\}$ and $TN^\perp \cap V = \{0\}$). As g is semi-Riemannian we have to analyse two subcases:

Case 2. I. 2. a. ($J(TN^\perp)$ is a *null vector bundle*). In this case $J(TN^\perp)$ and $J^3(TN^\perp)$ are degenerate distributions on N while $J^2(TN^\perp)$ is non-degenerate. Moreover, $\{J\xi, J^2\xi, J^3\xi\}$ is a set of linearly independent local vector fields for any $\xi \in \Gamma(TN^\perp)$. Therefore, the tangent bundle of N is decomposed as follows:

$$(2.2) \quad TN = \{J(TN^\perp) \oplus J^3(TN^\perp)\} \perp J^2(TN^\perp) \perp D_2.$$

It is important to note that D_2 is J -invariant. In particular, we state:

Proposition 1. *Let N be an orientable non-degenerate real hypersurface of a 4-dimensional e -metric ($J^4 = 1$)-manifold M such that $J(TN^\perp)$ is a null vector bundle. Then N is a parallelizable manifold.*

PROOF. It follows from the fact that ξ is globally defined and $\{J\xi, J^2\xi, J^3\xi\}$ is a set of linearly independent vector fields on N .

Case 2. I. 2. b. ($J(TN^\perp)$ is a *non-null vector bundle*). In this case $J^2(TN^\perp)$ is neither tangent nor normal to N at any point. However, $J(TN^\perp)$ and $J^3(TN^\perp)$ are distributions on N and we have the decomposition

$$(2.3) \quad TN = \{J(TN^\perp) \oplus J^3(TN^\perp)\} \perp D_3.$$

2. II. The case of a degenerate real hypersurface.

Now, suppose N is a degenerate real hypersurface of an e -metric ($J^4=1$)-manifold. Then $TN^\perp$ becomes a distribution on N and we have

$$(2.4) \quad TN = TN^\perp \perp SN,$$

where SN is a screen distribution on N . In this case we may use the e -metric structure (g, J) in order to get some particular distributions.

Case 2. II. 1. ($TN^\perp$ is a *vector sub-bundle* of U but $J(TN^\perp) \cap TN^\perp = \{0\}$). As in the non-degenerate case we have $J^2(TN^\perp) = TN^\perp$ and $J(TN^\perp) = J^3(TN^\perp)$. It follows that $J(TN^\perp)$ is a null distribution on N . Thus we can choose SN as a vector sub-bundle complementary of $TN^\perp$ in TN but such that $J(TN^\perp) \subset SN$. Then $J(tN)$ is a vector subbundle of SN too. Therefore we have the decomposition

$$(2.5) \quad TN = TN^\perp \perp \{J(TN^\perp) \oplus J(tN)\} \perp D_4,$$

where D_4 is the orthogonal distribution complementary of $J(TN^\perp) \oplus J(tN)$ in SN . Note that $\{\xi, J\xi, J\bar{\xi}\}$ is a set of linearly independent vector fields for any $\xi \in \Gamma(TN^\perp)$ and $\bar{\xi} \in \Gamma(tN)$ satisfying (1.3).

Case 2. II. 2. ($J\xi = \pm\xi, \forall \xi \in \Gamma(TN^\perp)$). In this case the decomposition is as in (2.4) and $J(tN)$ is not tangent to N .

Case 2. II. 3. ($TN^\perp \cap U = \{0\}$ and $TN^\perp \cap V = \{0\}$). We have to analyse the following two subcases:

Case 2. II. 3. a. ($J(TN^\perp)$ is a *null vector bundle*). Choose SN such that it contains the null distributions $J(TN^\perp)$, $J^2(TN^\perp)$ and $J^3(TN^\perp)$. It follows that $\xi, J\xi, J^2\xi$ and $J^3\xi$ are linearly independent on N and therefore $m > 2$. The tangent bundle of N can be decomposed as follows:

$$(2.6) \quad TN = TN^\perp \perp J(TN^\perp) \perp J^2(TN^\perp) \perp J^3(TN^\perp) \perp D_5.$$

Case 2. II. 3. b. ($J(TN^\perp)$ is a *non-null vector bundle*). It follows that $J(TN^\perp)$ and $J^3(TN^\perp)$ are non-null distributions on N while $J^2(TN^\perp)$ is a null vector bundle which is not tangent to N . Moreover $\xi, J\xi$ and $J^3\xi$ are linearly independent vector fields and therefore we have

$$(2.7) \quad TN = TN^\perp \perp J(TN^\perp) \perp J^3(TN^\perp) \perp D_6.$$

**§3. Integrability of distributions on a real
hypersurface of an e -($J^4 = 1$)-Kaehler manifold**

Suppose N is a non-degenerate real hypersurface of an e -($J^4 = 1$)-Kaehler manifold M such that TN decomposes as in (2.1). Then, taking into account that J is parallel with respect to $\bar{\nabla}$ and using (1.1), we obtain

$$(3.1) \quad J([X, Y]) = \nabla_X JY - \nabla_Y JX + \{h(X, JY) - h(Y, JX)\}\xi$$

for any $X, Y \in \Gamma(D_1)$. Applying J^3 to (3.1) and taking into account that $TN^\perp$ is a vector sub-bundle either of U or of V , we obtain

$$(3.2) \quad [X, Y] = J^3(\nabla_X JY - \nabla_Y JX) \pm \{h(X, JY) - h(Y, JX)\}J\xi.$$

It follows that $J^3(\nabla_X JY - \nabla_Y JX)$ is tangent to N . Moreover we have

$$g(J^3(\nabla_X JY - \nabla_Y JX), J\xi) = -g(\nabla_X JY - \nabla_Y JX, \xi) = 0,$$

and hence $J^3(\nabla_X JY - \nabla_Y JX)$ belongs to D_1 . Therefore from (3.2) we obtain:

Theorem 1. *Let N be a non-degenerate real hypersurface of an e -($J^4 = 1$)-Kaehler manifold M whose tangent vector bundle decomposes as in (2.1). Then the distribution D_1 is integrable iff the second fundamental form of N satisfies*

$$h(X, JY) = h(Y, JX), \quad \forall X, Y \in \Gamma(D_1).$$

In a similar way, using (1.1), (1.2) and the decompositions (2.2) and (2.3), we obtain the following characterizations of the integrability of distributions on N .

Theorem 2. *Let N be a non-degenerate real hypersurface of an e -($J^4 = 1$)-Kaehler manifold M whose tangent bundle decomposes as in (2.2). Then we have:*

(i) $J(TN^\perp) \oplus J^3(TN^\perp)$ is integrable iff h satisfies

$$h(J\xi, J\xi) + h(J^3\xi, J^3\xi) = 0$$

and

$$(3.3) \quad h(J\xi, J^3X) = h(JX, J^3\xi), \quad \forall X \in \Gamma(D_2);$$

(ii) $\{J(TN^\perp) \oplus J^3(TN^\perp)\} \perp J^2(TN^\perp)$ is integrable iff h satisfies

$$h(J\xi, JX) + h(J^3\xi, X) = 0, \quad h(J^3\xi, JX) + h(J^3\xi, X) = 0$$

and (3.3);

(iii) D_2 is integrable iff h satisfies

$$h(JX, JY) = h(J^2X, Y) = h(X, J^2Y), \quad \forall X, Y \in \Gamma(D_2).$$

Theorem 3. *Let N be a non-degenerate real hypersurface of an $e-(J^4 = 1)$ -Kaehler manifold M whose tangent bundle decomposes as in (2.3). Then we have:*

(i) $J(TN^\perp) \oplus J^3(TN^\perp)$ is integrable iff h satisfies

$$h(J\xi, J^3X) = h(J^3\xi, JX) = 0, \quad \forall X \in \Gamma(D_3);$$

(ii) D_3 is integrable iff h satisfies

$$h(X, JY) = h(Y, JX), \quad \forall X, Y \in \Gamma(D_3).$$

Next, we consider a totally umbilical non-degenerate real hypersurface N of M , i.e., the second fundamental form is expressed as follows:

$$h(X, Y) = \lambda g(X, Y), \quad \forall X, Y \in \Gamma(TN),$$

where λ is a differentiable function on N . If λ vanishes on N we say that N is totally geodesically immersed in M . Then, using Theorems 1, 2 and 3 we obtain the following results:

Corollary 1. *Let N be a totally umbilical non-degenerate real hypersurface of an $e-(J^4 = 1)$ -Kaehler manifold M whose tangent bundle admits the decomposition (2.1). Then D_1 is involutive iff N is totally geodesic.*

Corollary 2. *Let N be a totally umbilical non-degenerate real hypersurface of an $e-(J^4 = 1)$ -Kaehler manifold M whose tangent bundle decomposes as in (2.2). Then we have:*

(i) *The distributions $J(TN^\perp) \oplus J^3(TN^\perp)$ and $\{J(TN^\perp) \oplus J^3(TN^\perp)\}^\perp = J^2(TN^\perp)$ are integrable;*

(ii) *D_2 is integrable iff N is totally geodesic.*

Corollary 3. *Let N be a totally umbilical non-degenerate real hypersurface of an $e-(J^4 = 1)$ -Kaehler manifold M whose tangent bundle decomposes as in (2.3). Then we have:*

(i) *The distribution $J(TN^\perp) \oplus J^3(TN^\perp)$ is integrable;*

(ii) *If N is totally geodesic then D_3 is integrable. If N is not totally geodesic, D_3 is integrable iff D_3 and $J(D_3)$ are orthogonal distributions.*

We now consider a degenerate real hypersurface N of an $e-(J^4 = 1)$ -Kaehler manifold M whose tangent bundle satisfies (2.5). Using (1.3)–(1.8) we obtain

$$g([J\xi, J\bar{\xi}], \bar{\xi}) = C(J\xi, J\bar{\xi}) - B(J\bar{\xi}, J\bar{\xi})$$

and

$$g([J\xi, J\bar{\xi}], X) = C(J\xi, JX) - B(J\bar{\xi}, JX), \quad \forall X \in \Gamma(D_4).$$

Therefore we may state:

Theorem 4. *Let N be a degenerate real hypersurface of an e -($J^4=1$)-Kähler manifold M whose tangent bundle satisfies (2.5). Then we have:*

(i) $J(TN^\perp) \oplus J(tN)$ is integrable iff the second fundamental forms B and C satisfy

$$C(J\xi, J\bar{\xi}) = B(J\bar{\xi}, J\xi)$$

and

$$(3.4) \quad C(J\xi, JX) = B(J\bar{\xi}, JX), \quad \forall X \in \Gamma(D_4);$$

(ii) $TN^\perp \perp \{J(TN^\perp) \oplus J(tN)\}$ is integrable iff B and C satisfy (3.4) and

$$C(\xi, JX) = -B(J\xi, X) = -B(J\bar{\xi}, X), \quad \forall X \in \Gamma(D_4).$$

(iii) D_4 is integrable iff C is symmetric on D_4 and satisfies

$$C(X, JY) = C(Y, JX), \quad \forall X, Y \in \Gamma(D_4)$$

and B satisfies

$$B(X, JY) = B(Y, JX), \quad \forall X, Y \in \Gamma(D_4).$$

Similar results follow from decompositions (2.6) and (2.7). Therefore we may conclude that integrability of distributions on real hypersurfaces of e -($J^4 = 1$)-Kähler manifolds is characterized by means of the second fundamental forms.

Acknowledgements. The first author thanks European Community for financial support and the Consejo Superior de Investigaciones Científicas of Spain for their kind hospitality in Madrid.

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AUREL BEJANCU
TECHNICAL UNIVERSITY OF IAȘI
DEPARTMENT OF MATHEMATICS
6600 IAȘI
ROMANIA

GUSTAVO SANTOS-GARCÍA
DEPARTAMENTO DE FÍSICA APLICADA
FACULTAD DE CIENCIAS
PL. DE LA MERCED, S/N
37008 SALAMANCA
SPAIN

(Received September 17, 1993; revised January 7, 1994)